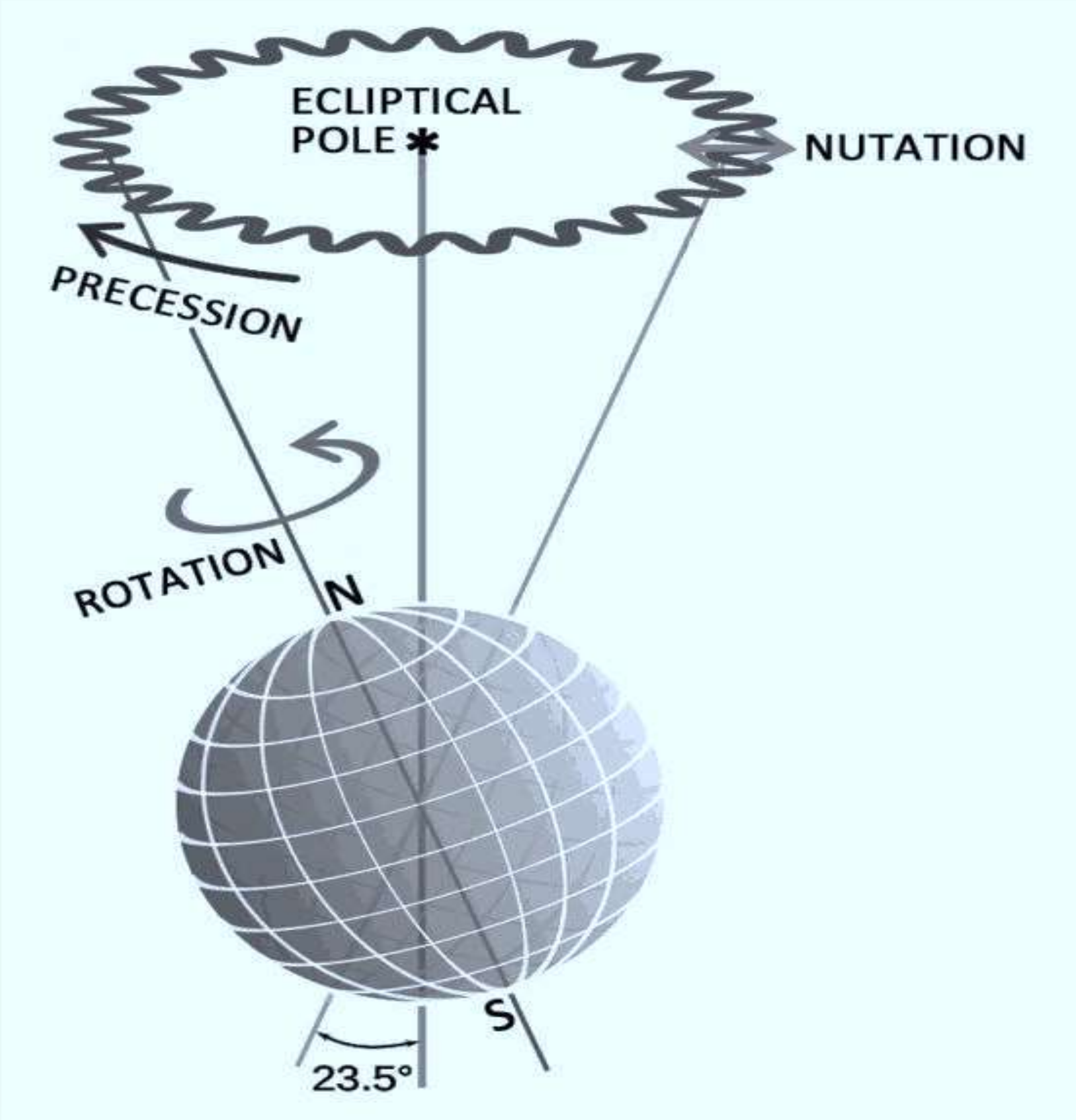
KARNATAKA STATE OPEN UNIVERSITY

Mukthagangothri, Mysuru – 570006

M.Sc. PHYSICS

(FIRST SEMESTER)



Course- MPDSC 1.1

Classical Mechanics

M.Sc. PHYSICS

FIRST SEMESTER

Course: MPDSC 1.1

CLASSICAL MECHANICS

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PRELUDE

Mechanics is the foundation of science and engineering. Its principles apply to vast range and variety of physical systems. The aim of classical mechanics is, and always will be, to understand physical phenomena and laws of mechanical world in its simplest form and to apply to diverse everyday situations. The behavior of classical systems is surprisingly rich. The word classical means time tested. This is the science which has been developed from ages starting from the invention of wheel from our ancestors to launching of mars mission recently. Still there is lot of scope of further growth.

This course aims in bringing a thorough understanding of classical mechanics and its techniques which are widely used in most of the branches of physics and engineering.

The course consists of four blocks, each block containing four units.

The first block starts with the basic understanding of Newtonian mechanics and the fundamental conservation laws of mechanics. Later it moves to analytical mechanics with the discussion of Lagrangian and its applications to some systems

In the second block we can see the further application of Lagrangian formulation to central force field in the first unit, introduction to scattering theory in the second and in later units the discussion of another alternative formulation, the Hamiltonian formulation in mechanics. Further in this block there are discussions on canonical transformations which form one of the very powerful tools in Classical Mechanics.

The third block starts with Poisson brackets, then Hamilton-Jacobi method which is considered to be powerful enough to bridge classical and quantum mechanics with few more modification favoring the quantum theory. Later units of this block provide a detailed discussion of rigid bodies.

The last block of this paper discusses very beautiful theory, the theory of relativity-special and general. With the addition of an extra dimension to the way of our thinking, it surprises us with its hard realities

UNIT-1: Fundamentals

Newton's laws of motion, frames of references, projectile motion with and without air resistance, Oscillations- Simple harmonic motion, damped and driven harmonic motions.

1.0 Objectives

After studying this unit you will be able to

- State and explain Newton's laws of motion
- Explain the need of frames of references
- Explain projectile motion with and without air resistance
- Explain the simple harmonic oscillations, damped and forced oscillations

1.1 Introduction

Mechanics is the description of motion of a system. The aim of mechanics is to find the position, velocity, acceleration, momentum, kinetic energy and many other properties associated with a physical system at any point of time. These variables are often expressed as functions of time and those expressions are called as equations of motion. Hence, we can equivalently say that the aim of the mechanics is to find the equations of motion of physical system.

1.2 Newton's laws of motion

Newton's laws of motion provide one of earliest descriptions of motion of particles, system of particles or any system in general. According to the description of Newtonian approach, a system under study is separated from everything else in the universe, that is also called surrounding. The system may get influenced by the surrounding. The sum total of these influences is called as force. Hence a force on a system is a mechanical influence on the system. Newton's laws of motion describe the motion of the system when there is no force acting on the system as well as when there is a force acting on the system. There are three laws of motion given by Newton, they can be termed as: Law of inertia; law of action; and law of action and reaction.

1.2.1 Law of inertia

The Newton's first law is also called law of inertia. It is defined as 'Every particle or a system continues to be in the state of rest or in uniform motion whenever there is no external force acting on the system'. According to Newton's first law of motion, an object at rest tends to stay at rest and an object in uniform motion (motion with constant velocity) tends to stay in the uniform motion with the same speed and in the same direction unless acted upon by an unbalanced force.

Objects often tend to resist changes in their state of motion. This tendency of opposing any changes in their states of motion is described as inertia. An object with higher inertia maintains its state of rest or uniform motion more effectively than an object with lesser inertia. For all translational motion, one can show that the inertia is equal to the mass. An object with higher mass has a greater tendency to resist changes in its state of motion and an object with lesser mass has lesser tendency to resist changes in its state of motion. Forces are said to be balanced when the net force on the object are zero. That is when all the forces are added up the resultant force is zero.

The first law defines the property of inertia. Hence it is also known as law of inertia. The Newton's first law as well as the property of inertia can be witnessed almost everyday in our surroundings.

A person sitting or standing in a bus tend to fall backward when the bus suddenly starts to move. This is due to inertia of rest. When the bus suddenly starts, the lower part of the body of the passenger which is in contact with the bus moves along with the bus while the upper part of the body tends to retain its state of rest due to inertia. As a result, the passenger falls backward. Similarly, when moving bus suddenly stops, the passengers sitting or standing in the bus are thrown forward due to inertia of motion. When a branch of a tree is vigorously shaken the fruits and seeds in it fall down due to inertia of rest.

1.2.2 Law of action

We defined the force as the mechanical influence of surroundings on a system. Many a times even though there are several forces acting on a particle or a system, there may be no net effect as they balance their effects. An influence in action can be observed only when the forces are unbalanced. The presence of an unbalanced forces influence the system to change its state of motion. Newton's second law describes motion of objects in the presence unbalanced forces. The Newton's second law states that 'The rate of change of momentum of a particle is directly proportional to the net force acting on the particle and it takes place along the direction of the force'.

$$\vec{F} = \frac{d\vec{p}}{dt}$$

The linear momentum is the product of mass and velocity. For most of the particles for which the mass can be treated as constant, the above expression becomes,

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt} = m\frac{d\vec{v}}{dt}$$

$$\vec{F} = m\vec{a}$$

Note that $\vec{F}$ here is the net force acting on the object. The net force is the sum of all the forces acting on an object. The first law refers to the simple case when the net external force on a body is zero. The second law of motion refers to the general situation when there is net external force acting on the body. It relates the net external force to the acceleration of the body. Thus, Newton's second law says the net force on a body is equal to the product of the body's mass and its acceleration.

The acceleration produced in an object is directly proportional to the force acting and inversely proportional to the mass of the object.

1.2.3 Law of action and reaction

Newton's third law describes the interaction between to particles or objects. During such an interaction a force acts on each body due to the other body. This implies there must be two objects participating in the interaction. One being pushed (or pulled) and another doing the pushing (or pulling). Thus, forces result from interactions between objects. These forces are always exist in pairs. Newton's third law describes these pair of forces and states as 'for every action there is an equal and opposite reaction'.

In every interaction, there is a pair of forces acting on the two interacting objects. Forces always come in pairs - equal and opposite action-reaction force pairs. For two interacting objects A and B we can write the Newton's third law,

$$\vec{F}_{AB} = -\vec{F}_{BA}$$

The negative sign indicate that these two forces are in opposite directions. We call the forces between two interacting bodies a third-law force pair. A third-law force pair exists when any two bodies interact in any situation. This is true if two objects are stationary or even if they are moving with constant velocity or with an acceleration.

1.3 Frames of references

A frame of reference includes the minimum necessary mathematical tools to describe the motion of a particle or a system of particles. It usually includes a coordinate system and a clock to describe where and when the particle is at any instant of time. There are two types of frames of references: An inertial frame of reference and a non inertial frame of reference.

A frame of reference in which Newton's laws of motion are valid are known as inertial frames of references. In such frames of references, an object changes its state of motion only when there is a net force acts on the object. Any frame of reference at rest and moving with constant velocity is a inertial frame of reference. Any frame moving with constant velocity with respect to an inertial frame of reference is also an inertial frame of reference.

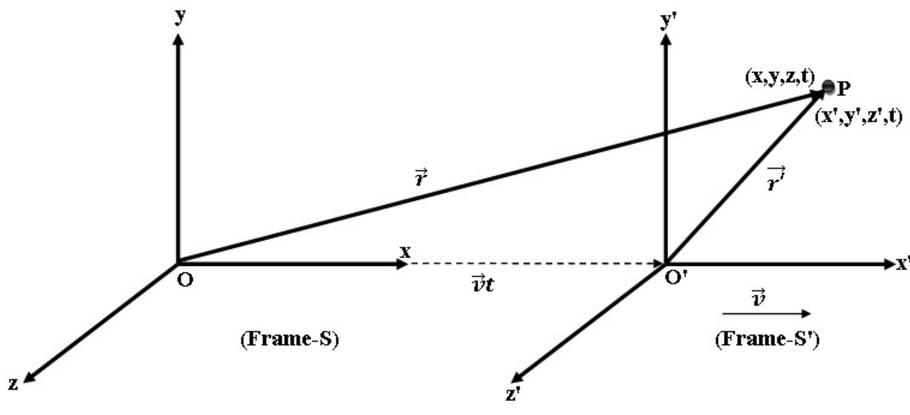

Figure 1.1: Inertial frames of reference

A frame of reference in which Newton's first law is not valid is known as non-inertial frame of reference. In such frames of references, an object may change its state of motion even when there is no unbalanced force acting on the object. An accelerated or a rotating frame of reference is a non-inertial frame of reference.

Consider two frames of references S and S' moving with relative velocity v along x-axis as shown in figure 1.1. The observes in the two frames will give two different coordinates to the same particle at a point P as observed by both. The coordinates are related to each other by transformation equations

$$x' = x - vt; \qquad y' = y; \qquad z' = z; \qquad t' = t$$

These equations are known as Galilean transformations.

During the description of motion of a particle or a system of particles when we use the term 'observer', it means with respect to a frame of reference fixed to the observer. More details about the frames of reference will be studied in Unit-13.

1.4 Projectile motion

A projectile motion is the motion of an object projected in uniform gravitational field like of earth's gravitational field on its surface. When an object is thrown at an angle of inclination with the horizontal, the path of the object is known as projectile.

1.4.1 Projectile motion without air resistance

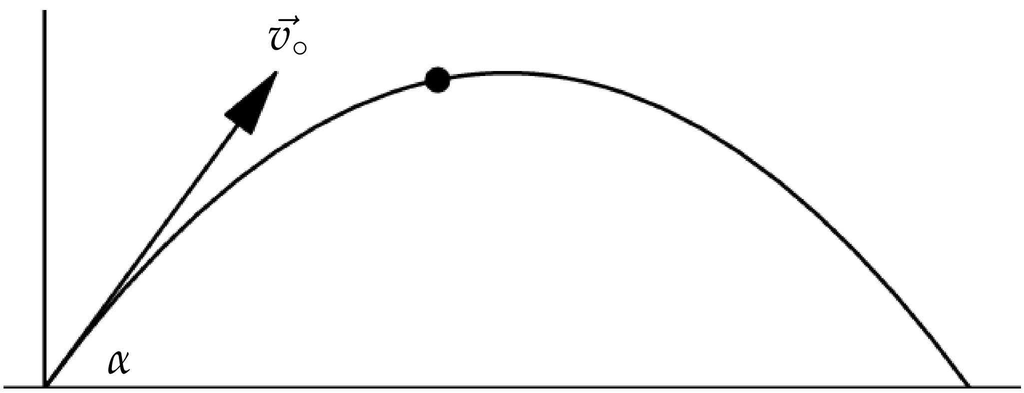

Figure 1.2: Projectile motion without air resistance

Let a body be projected at an angle α with the horizontal with a velocity $\vec{v}_\circ$. The motion will remain in the vertical plane of the velocity vector $\vec{v}_\circ$. Let us take the x-axis along the horizontal and the y-axis upward along the vertical direction in the plane of motion. We write the initial conditions as

$$x(t=0)=0; \qquad \text{and} \qquad y(t=0)=0$$

$$\dot{x}(t=0)=v_\circ \cos\alpha = U \qquad \text{and} \qquad \dot{y}(t=0)=v_\circ \sin\alpha = V$$

There is a uniform gravitational force acting on the body vertically downwards and there is no force acting on it along horizontal direction. Hence the equations of motion of the projectile will be

$$m\ddot{x}=0, \qquad \text{and} \qquad m\ddot{y}=-mg$$

Integrating the above equations we obtain

$$\dot{x}=C_1 \qquad \text{and} \qquad \dot{y}=-gt+C2$$

Using the initial conditions, we can show that $C_1=U$ and $C_2=V$. Then,

$$\dot{x}=U \qquad \text{and} \qquad \dot{y}=-gt+V$$

Upon integration and using the initial conditions again we get,

$$x=Ut \qquad \text{and} \qquad y=-\frac{1}{2}gt^2+Vt$$

The above two equations together provide the trajectory of the particle. we can combine them to obtain the equation of the path as $t=\dfrac{x}{U}$ and hence,

$$y=x\frac{V}{U}-\frac{1}{2}g\left(\frac{x}{U}\right)^2$$

We have

$$\frac{V}{U}=\tan\alpha$$

and

$$\frac{1}{U^2} = \frac{1}{{v_\circ}^2(\cos\alpha)^2} = \frac{1}{{v_\circ}^2}\sec^2\alpha = \frac{1}{{v_\circ}^2}(1+\tan^2\alpha)$$

Substituting these expressions in the equation of trajectory we get,

$$y = x\tan\alpha - \frac{gx^2}{2{v_\circ}^2}(1+\tan^2\alpha)$$

This is the equation of a parabola. Hence, when there is no air resistance, all projectile motion follow a parabolic path. We can define the maximum distance covered by the body as the range R, the expression for which can be shown to be

$$R = \frac{{v_\circ}^2\sin 2\alpha}{g}$$

The time required to cover this distance equal to range will be

$$T = \frac{2V}{g} = \frac{2v_\circ\sin\alpha}{g}$$

1.4.2 Projectile motion with air resistance

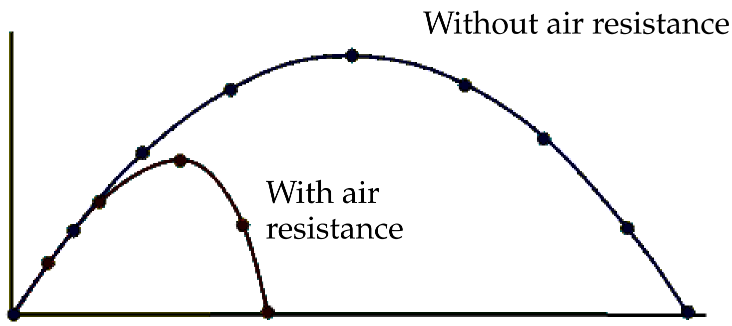

Figure 1.3: Projectile motion with air resistance

Consider the motion of a projectile in the atmosphere in which the retarding force is offered by the air resistance. Let us assume that the retarding force is proportional to the instantaneous velocity. In this case, the equations of motion of the projectile in the component form will be

$$m\ddot{x} = -km\dot{x} \qquad \text{and} \qquad m\ddot{y} = -mg - km\dot{y}$$

The first equation can be modified as

$$\ddot{x} = -k\dot{x} \implies \frac{d\dot{x}}{dt} = -k\dot{x} \implies \frac{d\dot{x}}{\dot{x}} = -k\,dt$$

Upon integration, we get,

$$\ln(\dot{x}) = -kt + C_1$$

If the initial velocity along x-direction at $t = 0$ is $v_\circ \cos\alpha = U$,

$$\ln(\dot{x}) = -kt + U \implies \dot{x} = Ue^{-kt}$$

Integrating once again to obtain the expression for x,

$$x = -\frac{U}{k}e^{-kt} + C_2$$

Using the initial condition that $x = 0$ when $t = 0$, we can show that $C_2 = \frac{U}{k}$. Then the definite solution to the above equation will be

$$x = \frac{U}{k}(1 - e^{-kt}) = \frac{v_\circ \cos\alpha}{k}(1 - e^{-kt})$$

Consider the second equation

$$m\ddot{y} = -mg - km\dot{y} \implies \ddot{y} = -g - k\dot{y} \implies \frac{d\dot{y}}{dt} = -g - k\dot{y}$$

$$\frac{d\dot{y}}{g + k\dot{y}} = -dt$$

On integration,

$$\frac{1}{k}\ln(g + k\dot{y}) = -t + C$$

The constant C can be obtained using the initial condition, $t = 0, \dot{y} = v_\circ \sin\alpha = V$. Therefore, $C = \frac{1}{k}\ln(g + kV)$. Then the above equation becomes,

$$\frac{1}{k}\ln(g + k\dot{y}) = -t + \frac{1}{k}\ln(g + kV) \implies \frac{1}{k}\ln(g + k\dot{y}) - \frac{1}{k}\ln(g + kV) = -t$$

$$ln\Big(\frac{g+k\dot{y}}{g+kV}\Big) = -kt \quad \Longrightarrow \quad \frac{g+k\dot{y}}{g+kV} = e^{-kt}$$

$$g+k\dot{y} = (g+kV)e^{-kt} \quad \Longrightarrow \quad \dot{y} = -\frac{g}{k} + \frac{g+kV}{k}\, e^{-kt}$$

Integrating once again,

$$y = -\frac{g}{k}t - \frac{(g+kV)}{k^2}e^{-kt} + C'$$

The value of the constant of integration C' can be obtain by the fact that $y = 0$ when $t = 0$ as $C' = \dfrac{g+kV}{k^2}$

Then the final solution to the equation will become,

$$y = -\frac{gt}{k} + \Big(\frac{g+kV}{k^2}\Big)(1-e^{-kt})$$

Thus the equations of the coordinates of the projectile as a functions of time will become

$$x = \frac{U}{k}(1-e^{-kt}) \quad \text{and} \quad y = -\frac{gt}{k} + \Big(\frac{g+kV}{k^2}\Big)(1-e^{-kt})$$

The path of the projectile is definitely not a parabola, but a much complex curve that depends on several parameters.

1.5 Oscillations

Oscillations are one of the fundamental motion types found in nature. Subatomic particles such as protons, neutrons, pions and even quarks execute oscillations. The oscillations are the basis of atomic and molecular spectra including IR, UV, NMR and many other spectroscopic investigations. The working of coupled electrical circuits are based on oscillations. Hence it is highly important to understand the oscillations in their fundamental description.

1.5.1 Simple harmonic motion

Simple harmonic motion is the simplest of all oscillatory motions. If a particle is executing oscillatory motion about a mean position x_o, if the instantaneous displacement of the particle at any point of time is x and instantaneous acceleration of the particle is a, then the oscillatory motion is said to be simple harmonic motion if

- The magnitude of acceleration at any point of time is always directly proportional to the displacement at that instant and
- The acceleration is always directed towards the mean position.

Mathematically, the motion is said to be simple harmonic if

$$a \, \alpha - x$$

The negative sign indicates that the acceleration is always directed towards mean position. The simple harmonic motion can also be defined in terms of restoring force that is responsible to maintain the particle at mean position that tries to bring it back to mean position whenever it is displaced from it as an oscillatory motion is said to be simple harmonic in nature if the restoring force that is always directed towards mean position is directly proportional to the instantaneous displacement of the particle.

$$F \, \alpha - x$$

A simple example of simple harmonic motion can be taken as the motion of a particle attached to a spring as shown in Figure 1.4.

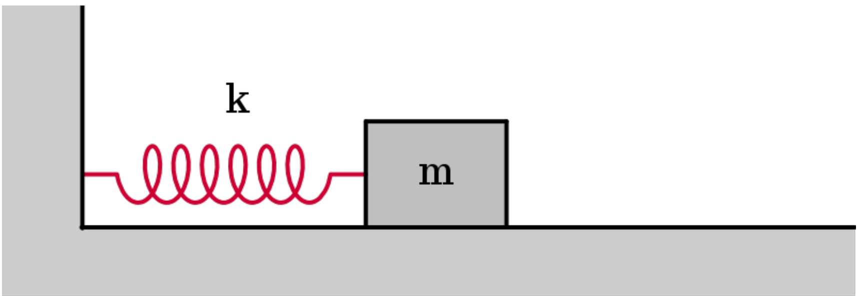

Figure 1.4: A mass attached to a spring executing simple harmonic oscillations

Let a mass m attached to a spring of spring constant k is displaced from its mean position. Whatever may be the direction of displacement, either an extension or compression is produced in the spring upon displacement that results in the development of a restoring force in the spring. We know form Hooke's law, the restoring force developed in the spring is proportional to the displacement of the mass.

$$\vec{F} \, \alpha - \vec{x}$$

We introduce the spring constant as the constant of proportionality in the above equation,

$$\vec{F} = -k\vec{x}$$

This is the well known Hooke's law. We also know that $F = ma = m\frac{d^2x}{dt^2}$, that gives,

$$m\frac{d^2x}{dt^2} = m\ddot{x} = -kx \implies \frac{d^2x}{dt^2} = \ddot{x} = -\frac{k}{m}\,x$$

Let, $\frac{k}{m} = \omega^2$, where $\omega = \sqrt{\frac{k}{m}}$ is the angular frequency of the simple harmonic oscillations. Then above equation reduces to

$$\ddot{x} = -\omega^2\,x \implies \ddot{x} + \omega^2 x = 0 \implies \left(\frac{d^2}{dt^2} + \omega^2\right) x = 0$$

The above equation is a second order homogeneous differential equation that represents the simple harmonic motion. One can confidently say any motion that satisfies above differential equation is known as simple harmonic motion. To find the solutions to above differential equation we have to determine the roots of auxiliary equation (that is $\lambda^2 + \omega^2 = 0$) that turns out to be $\pm i\omega$. The general solution to the above differential equation can be written as

$$x(t) = A\,e^{i\omega t} + Be^{-i\omega t}$$

Initial conditions determine the value of the constants of integration A and B in above equation. Let us assume the particle was at extreme position with an amplitude of x_o when $t = 0$ and was momentarily at rest, that means the velocity was zero at its extreme position.

Then applying the initial conditions to the general solution, we get,

$$x = A\,e^{i\omega t} + Be^{-i\omega t} \implies x_o = Ae^0 + Be^0 \implies x_o = A + B$$

$$\dot{x} = A\,e^{i\omega t}\,(i\omega)\; + Be^{-i\omega t}(-i\omega) \implies 0 = Ae^0(i\omega) - Be^0(i\omega) \implies 0 = A - B$$

Solving the above two equations gives us, $A = \frac{x_o}{2}$ and $B = \frac{x_o}{2}$. Using these values of constants in the general solution we can rewrite the expression for instantaneous displacement of the particle as

$$x = \frac{x_o}{2}\,e^{i\omega t} + \frac{x_o}{2}e^{-i\omega t} \;=\; x_o\frac{e^{i\omega t} + e^{-i\omega t}}{2} = x_o \cos \omega t$$

Note that we got the instantaneous displacement expression as

$$x(t) = x_o \cos \omega t$$

If the initial conditions had been chosen as particle being at mean position when $t = 0$ and velocity being maximum, then we would have got

$$x(t) = x_o \sin \omega t$$

as the solution. Note that these are harmonic functions. Because the instantaneous displacement of a particle executing this kind of motion is represented by simplest harmonic function (single sine or cosine or even combination of them, but single frequency), the motion is called simple harmonic motion. A sample representation of instantaneous displacement of a particle executing simple harmonic oscillation is provided in Figure 1.5. Hence, one can also define simple harmonic motion as one in which the instantaneous displacement is either sine or cosine function of time.

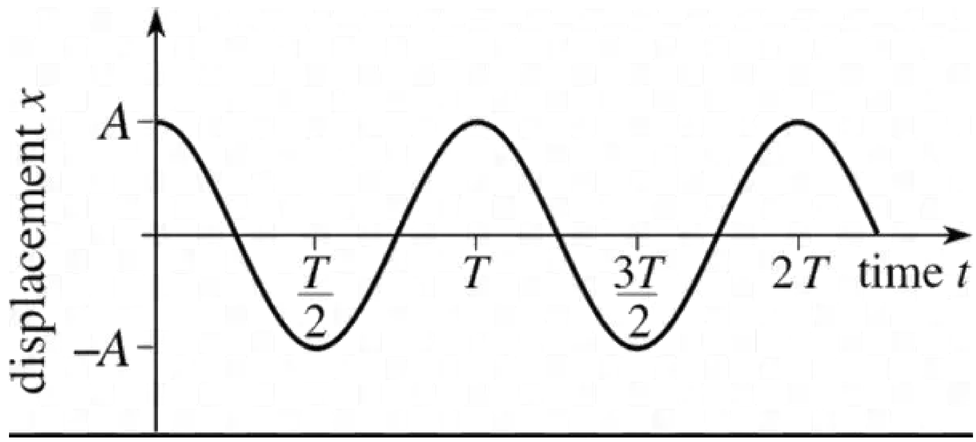

Figure 1.5: Sinusoidal representation of the motion of a particle executing simple harmonic oscillations

1.5.2 Damped oscillations

Suppose that a resistive or a damping force is present in addition to the restoring force which is necessary to produce the oscillations. We assume that the damping force is proportional to the velocity of the particle and is given by $F_{damp} = -2m\mu\dot{x}$ with positive value of μ. The quantity $2m\mu$ represents the damping force per unit velocity and is so chosen to simplify the calculations. The equation of motion, then becomes

$$m\ddot{x} = -kx - 2m\mu\dot{x} \qquad \Longrightarrow \qquad \ddot{x} + 2\mu\dot{x} + \omega_\circ^2 x = 0$$

Where, $\omega_\circ^2 = \dfrac{k}{m}$ as before.

The above equation is a second order homogeneous linear differential equation whose solutions can be determined by finding the roots of its auxiliary equation that is $\alpha^2 + 2\mu\alpha + \omega_\circ^2 = 0$

The two roots of the auxiliary equation will be $\alpha = -\mu \pm \sqrt{\mu^2 - \omega_\circ^2} = \mu \pm \lambda$. With $\lambda = \sqrt{\mu^2 - \omega_\circ^2}$. Then the solution of the above differential equation can be written as

$$x = Ae^{-(\mu+\lambda)t} + Be^{-(\mu-\lambda)t} = e^{-\mu t}(Ae^{-\lambda t} + Be^{+\lambda t})$$

This is the general solution for the a damped oscillator. Depending on the values of μ and $\omega_\circ$, different situations arises.

Case I: Over-damped motion

If $\mu > \omega_\circ$, we can observe that $\mu > \lambda$ and λ is a positive number. Then the solution turns out to be product of an exponential decay term $e^{-\mu t}$ and a combination of an exponential decay and growth terms $(Ae^{-\lambda t} + Be^{+\lambda t})$.

With an initial condition of $x = x_\circ$ and $\dot{x} = v_\circ$ at $t = 0$, we can show that

$$A = -\frac{v_\circ + (\mu - \lambda)x_\circ}{2\lambda} \quad \text{and} \quad B = \frac{v_\circ + (\mu + \lambda)x_\circ}{2\lambda}$$

When both x_o and v_o are positive, B, the coefficient of growth term is positive and A, the coefficient of decay is negative. Moreover, the magnitude of B is greater than that of A.

If on the other hand, v_o is negative, such that $v_o < -(\mu + \lambda)x_o$, B will be negative and A will be positive. Furthermore, magnitude-wise A is greater than B. Since the term containing A decays more rapidly than the term containing B, the term containing B will be predominant after some time when the term containing A becomes insignificant. Thus, the positive displacement will become negative crossing the equilibrium position and once again will tend monotonically to the equilibrium position as shown in the figure 1.6.

Case II: Critically damped motion

If $\mu = \omega_\circ$, we can observe that
$lambda = 0$. Then expression for the displacement will be

$$x = (A + B)e^{-\mu t}$$

Note that the solution turns out to be an exponential decay because of the presence of the term $e^{-\mu t}$. Hence under this condition the displacement of the particle decreases exponentially with time towards mean position.

Case III: Under-damped motion

If $\mu < \omega_\circ$, we can observe that λ becomes an imaginary number. Let us call it as $\omega' = \sqrt{\omega_o^2 - \mu^2}$. Then the solution of the oscillator becomes

$$x = e^{-\mu t}(Ae^{i\omega' t} + Be^{-i\omega' t})$$

The terms $e^{i\omega' t}$ and $e^{-i\omega' t}$ are essentially oscillatory functions. Hence the motion will be oscillatory. However, the presence of exponential decay term $e^{-\mu t}$ indicates that the amplitude of the oscillations decreases exponentially. The variation of displacement of a particle executing damped oscillations with under-damped condition is shown in the figure 1.6.

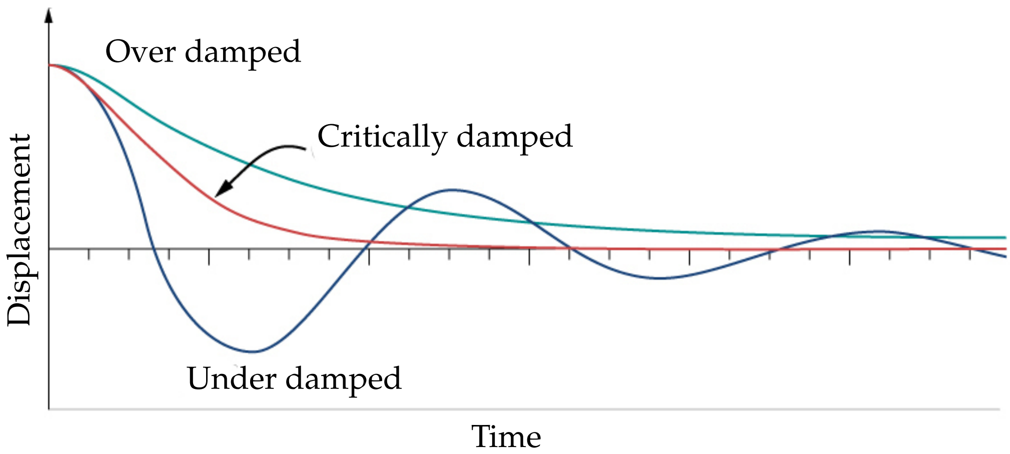

Figure 1.6: Representation of under damped, critically damped and over damped motion

1.5.3 Forced oscillations

We saw in the previous section that a damped system would oscillate under certain conditions. The amplitude of oscillation goes on decreasing since the energy is dissipated in overcoming the resistive force. If the oscillations are to be maintained, energy must be supplied to the system to make up for the losses. If this is done by applying an external driving force which is time dependent, the oscillations are called the forced oscillations. The equation of motion of such a system is written as

$$m\ddot{x} = -kx - 2m\mu\dot{x} + F(t) \qquad \Longrightarrow \qquad \ddot{x} + 2\mu\dot{x} + \omega_\circ^2 x = \frac{F(t)}{m}$$

Let us assume the driving force is sinusoidal as $F(t) = F_o \sin\omega t$

Then the equation of motion of the particle executing the forced oscillations will become

$$\ddot{x} + 2\mu\dot{x} + \omega_\circ^2 x = \frac{F_o \sin\omega t}{m}$$

This is a linear inhomogeneous differential equation and can be solved in may different ways. The simplest method is to assume a solution as $x = A\sin(\omega t - \theta)$ and determine the constants A and θ in accordance with the differential equation and the initial conditions.

Substituting the $x = A\sin(\omega t - \theta)$, $\dot{x} = A\cos(\omega t - \theta)\,\omega$ and $\ddot{x} = -A\sin(\omega t - \theta)\,\omega^2$ in the above differential equation we get

$$-A\omega^2 \sin(\omega t - \theta) + w\mu A\omega\cos(\omega t - \theta) + \omega_o^2 A\sin(\omega t - \theta) = \frac{F_o}{m}\sin\omega t$$

The right hand term in the above equation can be modified by adding and subtracting θ as

$$\frac{F_o}{m}\sin\omega t = \frac{F_o}{m}\sin\big((\omega t - \theta) + \theta\big) = \frac{F_o}{m}\sin(\omega t - \theta)\cos\theta + \frac{F_o}{m}\cos(\omega t - \theta)\sin\theta$$

Therefore

$$-A\omega^2 \sin(\omega t - \theta) + w\mu A\omega\cos(\omega t - \theta) + \omega_o^2 A\sin(\omega t - \theta)$$

$$= \frac{F_o}{m}\sin(\omega t - \theta)\cos\theta + \frac{F_o}{m}\cos(\omega t - \theta)\sin\theta$$

Comparing the coefficients of $\sin(\omega t - \theta)$ and $\cos(\omega t - \theta)$ on the two sides of the above equation, we can write,

$$-A\omega^2 + \omega_o^2 A = \frac{F_o}{m}\cos\theta \qquad \text{and} \qquad 2\mu A\omega = \frac{F_o}{m}\sin\theta$$

On solving the above two equations we can get the expressions for A and θ as

$$A = \frac{F_o/m}{\sqrt{(\omega_o^2 - \omega^2)^2 + 4\mu^2\omega^2}} \qquad \text{and} \qquad \theta = \tan^{-1}\left(\frac{2\mu\omega}{(\omega_o^2 - \omega^2)}\right)$$

With these constants, we can write the expression for the instantaneous position of the particle executing the forced oscillations as

$$x = \frac{F_o/m}{\sqrt{(\omega_o^2 - \omega^2)^2 + 4\mu^2\omega^2}}\sin(\omega t - \theta)$$

1.5.4 Resonance

Resonance is a condition in forced oscillations in which the amplitude of the oscillations increases and becomes maximum at certain frequency of the periodic force applied.

This maximum point can be determined using the expression for amplitude and investigating for the maximum of the function.

$$A = \frac{F_o/m}{\sqrt{(\omega_o^2 - \omega^2)^2 + 4\mu^2\omega^2}}$$

The amplitude becomes maximum when $\dfrac{dA}{d\omega} = 0$ and $\dfrac{d^2A}{d\omega^2} < 0$. Hence at maximum,

$$\frac{dA}{d\omega} = \frac{d}{d\omega}\left[\frac{F_o}{m}\left((\omega_o^2 - \omega^2) - 4\mu^2\omega^2\right)^{-1/2}\right] = 0$$

$\Longrightarrow$

$$\frac{-F_o}{2m}\left[\left((\omega_o^2 - \omega^2)^2 + 4\mu^2\omega^2\right)^{-3/2}\left(2(\omega_o^2 - \omega^2)\cdot 2\omega + 4\mu^2\cdot 2\omega\right)\right] = 0$$

$\Longrightarrow$

$$\frac{-F_o\left(2(\omega_o^2 - \omega^2)\cdot 2\omega + 4\mu^2\cdot 2\omega\right)}{2m\left((\omega_o^2 - \omega^2)^2 + 4\mu^2\omega^2\right)^{3/2}} = 0$$

$\Longrightarrow$

$$\left(2(\omega_o^2 - \omega^2) \cdot 2\omega + 4\mu^2 \cdot 2\omega\right) = 0$$

$$\implies$$

$$2(\omega_o^2 - \omega^2) + 4\mu^2 = 0 \quad \implies \quad \omega^2 = \omega_o^2 - 2\mu^2$$

Hence the amplitude of the oscillation becomes maximum when the frequency of the periodic driving force becomes $\omega = \sqrt{\omega_o^2 - 2\mu^2}$. This is called resonance.

Further, we can propose that the damping is very less and $\omega_o^2 >> 2\mu^2$ under which the resonance will occur when $\omega = \omega_o$. In words, The amplitude becomes maximum when the frequency of the driving force becomes equal to the natural frequency of the oscillator.

1.6 Check your progress

Check your progress by answering the questions below.

1. What is inertia?
2. Newton's third law says that for every action there is ______________.
3. What is an inertial frame of reference?
4. What is a simple harmonic oscillation?
5. Explain the phenomena of resonance.

1.7 Keywords

- Inertia
- Frame of reference
- Newtonian mechanics
- Free and forced oscillations

1.8 Worked examples

1. A stone of mass 200 g is thrown in a direction that makes 60 degree angle with the horizontal with an initial velocity 30 ms^{-1}. Determine the range of the stone assuming there is no air friction.

   Answer:

   Data:

   $$m = 200g = 0.2kg$$

   $$v_o = 30ms^{-1}$$

   $$\theta = 60^o$$

   We know the expression for the range of the projectile

   $$R = \frac{v_o^2 \sin 2\theta}{g}$$

   $$R = \frac{30^2 \sin 120}{9.8} = 79.53m$$

2. A spherical metal ball is thrown with an initial velocity of 40 ms^{-1} along 45^o with horizontal plane. Assuming the acceleration due to gravity as 9.8 ms^{-2s} and coefficient of resistance as 0.5, determine the range of the ball. Compare the value when there is no air friction.

   Answer:

   Data:

   $$v_o = 40ms^{-1}$$

   $$g = 9.8ms^{-2}$$

   $$\theta = 45^o$$

   $$k = 0.5$$

   Because there is no direct expression for the range, we should find by the fact that the vertical displacement becomes zero upon reaching the horizontal plane. Using the following function we shall calculate the time of flight and the horizontal distance covered with that time of flight would be the range of the ball.

$$x = \frac{U}{k}(1 - e^{-kt}) \quad \text{and} \quad y = -\frac{gt}{k} + \Big(\frac{g + kV}{k^2}\Big)(1 - e^{-kt})$$

If $y = 0$, we get,

$$\frac{gt}{k} = \Big(\frac{g + kV}{k^2}\Big)(1 - e^{-kt})$$

$$\frac{9.8t}{0.5} = \Big(\frac{9.8 + 0.5 \times 40 \times \sin 45}{0.5^2}\Big)(1 - e^{-0.5t})$$

$$19.6t = 95.76(1 - e^{-0.5t})$$

Expanding the exponential term upto second order term (This is just an approximation. Higher order terms provide higher accuracy)

$$19.6t = 95.76\big[1 - (1 - 0.5t + \frac{0.5^2t^2}{2})\big]$$

On solving the above equation, we get, $t = 2.3625s$

To find the range of the ball, let us find the horizontal distance travelled in this duration of time,

$$x = \frac{U}{k}(1 - e^{-kt}) = \frac{40 \times \cos 45}{0.5}(1 - e^{-0.5 \times 2.3625}) = 39.592m$$

In the absence of the air friction, the range would have been

$$R = \frac{40^2 \sin 90}{9.8} = 163.26m$$

Note that how much reduction in the range is caused by the air friction.

3. A sphere of mass 200 g is attached to a massless spring of spring constant 3.2 Nm^{-1} is set into simple harmonic oscillations with with an amplitude of 8 cm. If there is a damping coefficient of 0.5 is present due to air friction, check whether the oscillator is over damped or critically damped or under damped. If under damped also determine the frequency of oscillations.

   Answer: We know

$$\omega_o = \sqrt{\frac{k}{m}} = \sqrt{\frac{3.2}{0.2}} = \sqrt{16} = 4\text{second}$$

Given $\mu = 0.5$. This is less than ω_o, Therefore the situation in under damped. The frequency of oscillation will be

$$\omega' = \sqrt{\omega_o^2 - \mu^2} = \sqrt{4^2 - 0.5^2} = \sqrt{15.75} = 3.9686Hz$$

4. A mass of 250 g attached to a massless spring resonates at 3.2 Hz. If there is a damping with coefficient of 1.2, determine the spring constant.

   Answer: We know the expression for the resonance frequency as

$$\omega^2 = \omega_o^2 - 2\mu^2 \qquad \Longrightarrow \qquad \omega_o^2 = \omega^2 + 2\mu^2$$

$$\omega_o^2 == \frac{k}{m} = 3.2^2 + 2 \times 1.2^2 = 13.12$$

$$\therefore k = 13.12 \times 0.25 = 3.28Nm^{-1}$$

1.9 Questions for self study

1. State and explain Newton's laws of motion with examples.
2. What are inertial and non inertial frames of references?
3. Describe the motion of a projectile without air resistance and determine the range and time of flight.
4. Describe the motion of a projectile with air friction. Determine the horizontal and vertical distance travelled as a function of time.
5. What are simple harmonic oscillations? Determine the expression for the instantaneous displacement of a simple harmonic oscillator.
6. What are damped oscillations? Provide the theory of damped oscillations.
7. Describe the forced oscillations and explain the phenomena of resonance.

1.10 Answers to check your progress

1. Inertia is a property that is responsible to maintain the state of motion of a body.
2. Newton's third law says that for every action there is an equal and opposite reaction.
3. A frame of reference in which Newton's laws are valid is called an inertial frame of reference.
4. An oscillation motion in which the instantaneous acceleration is always proportional to the displacement and directed towards the mean position is called simple harmonic oscillation.
5. Under forced oscillations, when the frequency of the driving force becomes equal to the natural frequency of the oscillating system, the amplitude of the oscillation becomes maximum. This phenomena is known as resonance.

1.11 References

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UNIT-2: Conservation Laws

Mechanics of a particle and system of particles, conservation of energy and momentum, constraints, generalized coordinates.

2.0 Objectives

After studying this unit you will be able to

- Describe the mechanics of a single particle.
- State and prove conservation of linear and angular momentum of single and a system of particles.
- State and prove conservation of energy of a single and system of particles.
- Define the center of mass and describe the dynamics of center of mass.
- Describe the constraints of motion and their classification.
- Describe the need for generalized coordinates.

2.1 Introduction

Mechanical description of any physical system that consists of a large number of particles can be easily understood when the mechanics of a single particle is known. In this section we shall understand the mechanics of single particle in Newtonian approach.

2.2 Mechanics of a single particle

Let $\vec{r}$ be the position vector of a particle of mass m. Then the velocity of the particle will be

$$\vec{v} = \frac{d\vec{r}}{dt}$$

The linear momentum of the particle will be

$$\vec{p} = m\,\vec{v} = m\,\frac{d\vec{r}}{dt}$$

As the particle gets influenced by the external objects and fields which we together call as surrounding, the particle may experience various forces. The vector sum of these forces will be non zero when they are not balanced. Let $\vec{F}$ denote the net external force acting on the particle. Then from Newton's second law,

$$\vec{F} = \frac{d\vec{p}}{dt}$$

When the force acting on the particle is known as a function of time, above equation provides a means of finding the equations of motion of the particle.

$$m\,\frac{d\vec{v}}{dt} = \vec{F}(t) \qquad \Longrightarrow \qquad m\,\frac{d^2\vec{r}}{dt^2} = \vec{F}(t)$$

$$m\,\frac{d^2\vec{r}}{dt^2} - \vec{F}(t)$$

The above equation is a differential equation that represents the equation of motion of the particle. If the force acting on the particle is known as a definite function of time and if the initial conditions of the particle are known one can solve the above differential equation through integration and obtain the expression for $\vec{r}$ as function of time and hence can determine the position of the particle at any instant. When the position function is known, by differentiating it we can determine the velocity. Further, all the physical properties associated with the particle can be calculated using those equation.

2.2.1 Conservation of linear momentum

If $\vec{F} = 0$, then we get,

$$\frac{d\vec{p}}{dt} = 0 \qquad \Longrightarrow \qquad \vec{p} = constant$$

Hence, when the net external force acting on the particle is zero, the linear momentum gets conserved. This is known as conservation of linear momentum.

This is also another representation of Newton's first law which says when there is no force acting on the particle it continues to be in its state of rest or motion with constant velocity.

Motion with constant velocity also implies motion with constant linear momentum.

2.2.2 Conservation of angular momentum

The angular momentum of the particle with respect to a point O (This point O is the origin of the frame of reference used to describe the motion) is defined as

$$\vec{L} = \vec{r} \times \vec{p}$$

If there is a force $\vec{F}$ acting on the particle, we can define the moment of the force, or the torque with respect to the point O as

$$\vec{N} = \vec{r} \times \vec{F}$$

Consider the rate of change of angular momentum

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \vec{r} \times \frac{d\vec{p}}{dt} + \frac{d\vec{r}}{dt} \times \vec{p}$$

$$\frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt} + \vec{v} \times m\,\vec{v}$$

The second term consists of cross product between two parallel vectors and hence becomes zero. Hence,

$$\frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt}$$

From Newton's second law we know $\frac{d\vec{p}}{dt} = \vec{F}$. Therefore,

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \vec{N}$$

Thus the rate of change of angular momentum is equal to the torque acting on the particle. Hence, if the total external torque acting on the particle becomes zero, then the angular momentum remains constant.

Thus the conservation of angular momentum states that, when the total external torque acting on a particle becomes zero, the angular momentum of the particle gets conserved.

2.2.3 Conservation of energy

During the course of motion, let us assume the particle moves from position 1 to position 2 under the influence of the external force. Then the work done by the force upon this motion will be the integral of dot product between the force $\vec{F}$ and displacement $d\vec{s}$.

$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{s}$$

If we assume the mass of the particle as constant, the above integral can be written as

$$\int \vec{F} \cdot d\vec{s} = \int \frac{d\vec{p}}{dt} \cdot \vec{v} dt = m \int \frac{d\vec{v}}{dt} \cdot \vec{v} dt$$

We can show that $\frac{d\vec{v}}{dt} \cdot \vec{v} = \frac{1}{2}\frac{d}{dt}(v^2)$. Then the work done expression will become,

$$W_{12} = \frac{m}{2}\int_1^2 \frac{d(v^2)}{dt}\, dt = \frac{m}{2}\int_1^2 d(v^2) = \frac{m}{2}(v_2^2 - v_1^2)$$

$$W_{12} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 = T_2 - T_1 \tag{2.1}$$

Thus we can observe that the work done in moving the particle from position 1 to position 2 is equal to the difference in the kinetic energy of the particle at the two positions.

If the work done by the force field to take the particle from position 1 to position 2 along any possible path between the two position remains same, then the force is said to be conservative in nature. In other words, if the total work done in taking the particle from position 1 to position 2 along a path and to bring back the particle to position 1 along a different path is zero, then the force field is said to be conservative.

$$\oint \vec{F} \cdot d\vec{s} = 0$$

If the force field is conservative in nature, the force can be expressed as the negative gradient of a scalar function. This scalar function is called potential.

$$\vec{F} = -\vec{\nabla} V(\vec{r})$$

The work done by the force can be taken as

$$\vec{F} \cdot d\vec{s} = -\vec{\nabla} V(\vec{r}) \cdot d\vec{s} = -dV$$

Then the work done in taking the particle from position 1 to position 2 would be

$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{s} = -\int_1^2 dV$$

$$W_{12} = -(V_2 - V_1) = V_1 - V_2 \tag{2.2}$$

From equation 2.1 and 2.2 we can conclude that,

$$T_2 - T_1 = V_1 = V_2 \qquad \Longrightarrow \qquad T_1 + V_1 = T_2 + V_2$$

Thus we can conclude that the sum of kinetic and potential energies at position 1 and position 2 are equal to each other. Generalizing the argument we can show that the sum of kinetic and potential energies, which we often call as total mechanical energy is constant at every point along the path of the particle. Hence, when the forces acting on the particle are conservative in nature, the total energy remains constant at every point of the motion.

2.3 Mechanics of a system of particles

Consider a system of N number of particles. Let m_i and $\vec{r}_i$ be the mass and position of i^{th} particle respectively. Let $\vec{F}^{ext}$ be the external force acting on the system. Let $M = \Sigma_{i=1}^{N} m_i$ be the total mass of the system. We define center of mass of the system as a point where the total mass can be assumed to be concentrated. It is also the point about which the mass distribution is symmetrical. The position vector of the center of mass is defined as

$$\vec{R} = \frac{\sum_i^N m_i r_i}{\sum_i^N m_i} = \frac{1}{M} \Sigma m_i r_i \tag{2.3}$$

Each particle in the system experiences two forces: an external force and an internal force due to all other particles. Let $\vec{F}_i^{ext}$ be the external force on i^{th} particle and $\vec{F}_{ij}^{int}$ be the internal force on i^{th} particle due to j^{th} particle. Then the total force acting on the i^{th} particle would be

$$\vec{F}_i^{tot} = \vec{F}_i^{ext} + \sum_j \vec{F}_{ij}^{int}, \quad i \neq j \tag{2.4}$$

From Newton's second law we know that the rate of change of momentum is equal to the total force acting on the system. Hence,

$$\frac{d\vec{p}}{dt} = \dot{\vec{p}} = \vec{F}_i^{ext} + \sum_j \vec{F}_{ij}^{int}, \quad i \neq j \tag{2.5}$$

The first term on the right hand side of the equation 2.5 represents the external force on the i^{th} particle and the second term is the vector sum of all the internal forces due to the ineraction of the remaining $N-1$ particles with i^{th} particle. From Newton's third law we should remember that these interaction forces are equal and opposite. That means, $F_{ij}^{int} = -F_{ji}^{int}$.

The total rate of change of momentum of the whole system can be taken by taking the summation over i in equation 2.5

$$\sum_i^N \dot{\vec{p}}_i = \sum_i^N \vec{F}_i^{ext} + \sum_{i,j}^N \vec{F}_{ij}^{int} \tag{2.6}$$

In the above expression the second term in the right hand side consists of terms that come in pair wise. For example, if there is a term $\vec{F}_{13}^{int}$, there will also be a term $\vec{F}_{31}^{int}$. From Newton's third law, we know that the sum of these two terms and hence all the pairs become zero as the interaction forces are equal and opposite. hence, the second term in the right hand side of the above equation becomes zero.

$$\sum_i^N \dot{\vec{p}}_i = \sum_i^N \vec{F}_i^{ext} \tag{2.7}$$

$$\sum_i^N \frac{d\vec{p}_i}{dt} = \sum_i^N m_i \frac{d\vec{v}_i}{dt} = \sum_i^N m_i \frac{d^2\vec{r}_i}{dt^2} = \sum_i^N \vec{F}_i^{ext} \tag{2.8}$$

$$\frac{d^2}{dt^2} \sum_i^N m_i \vec{r}_i = \sum_i^N \vec{F}_i^{ext} \tag{2.9}$$

From the definition of center of mass we know $\sum_i^N m_i \vec{r}_i = M\vec{R}$. Hence, we can write

$$M \frac{d^2\vec{R}}{dt^2} = M\ddot{\vec{R}} = \vec{F}^{ext} \tag{2.10}$$

2.3.1 Conservation of linear momentum

The total linear momentum of the system can be taken as the sum of the linear momentum of each particle.

$$\sum_i^N m_i \,\dot{\vec{r}}_i = \frac{d}{dt}\sum_i^N m_i \vec{r}_i = M\,\dot{\vec{R}} \tag{2.11}$$

Hence, we can say that the linear momentum of center of mass is the linear momentum of the whole system of particles. From equation 2.12 we can show that when the total external force acting on the system becomes zero,

$$M\,\frac{d^2\vec{R}}{dt^2} = M\ddot{\vec{R}} = 0 \qquad \Longrightarrow \qquad M\,\frac{d\vec{R}}{dt} = M\,\dot{\vec{R}} = constant \tag{2.12}$$

This is nothing but the conservation of linear momentum of the system of particles. Hence we can state the law of conservation of linear momentum of system of particles as 'When to total external force acting on the system becomes zero, then the total linear momentum of the system remains conserved'. Further, we can observe that the center of mass moves with constant velocity when the total external force is zero.

2.3.2 Conservation of angular momentum

The total angular momentum of the system about any point will be equal to the vector sum of the angular momenta of individual particles. Let $\vec{l}_i$ represent the angular momentum of the i^{th} particle about some point. Then,

$$\vec{l}_i = \vec{r}_i \times \vec{p}_i$$

In the above equation $\vec{r}_i$ is the position vector of i^{th} particle from the given point. Hence the total angular momentum of the system of particle can be obtained by taking the vector sum of the individual momenta as

$$\vec{L} = \sum_i^N \vec{l}_i = \sum_i^N \vec{r}_i \times \vec{p}_i$$

Let $\vec{N}$ be the total torque acting on the system, then,

$$\vec{N} = \frac{d\vec{L}}{dt} = \frac{d}{dt}\sum_i^N \vec{r}_i \times \vec{p}_i = \sum_i^N \dot{\vec{r}}_i \times \vec{p}_i + \sum_i^N \vec{r}_i \times \dot{\vec{p}}_i$$

In the first term, $\dot{\vec{r}}_i \times \vec{p}_i = \dot{\vec{r}}_i \times \dot{\vec{r}}_i = 0$

Further, $\sum_i^N \vec{r}_i \times \dot{\vec{p}}_i = \sum_i^N \vec{r}_i \times (\vec{F}_i^{ext} + \sum_j^N \vec{F}_{ij}^{int})$

Consider

$$\sum_i^N (\vec{r}_i \times \sum_j^N \vec{F}_{ij}^{int}) = \sum_{ij}^N \vec{r}_i \times \vec{F}_{ij}^{int}$$

In the above expression, the terms always come in pairs. Hence one can write,

$$\sum_{ij}^N \vec{r}_i \times \vec{F}_{ij}^{int} = \frac{1}{2}(\sum_{ij}^N \vec{r}_i \times \vec{F}_{ij}^{int} + \sum_{ij}^N \vec{r}_j \times \vec{F}_{ji}^{int})$$

From Newton's third law, we know that $\vec{F}_{ij} = -\vec{F}_{ji}$, Hence,

$$\sum_{ij}^N \vec{r}_i \times \vec{F}_{ij}^{int} = \frac{1}{2}(\sum_{ij}^N \vec{r}_i \times \vec{F}_{ij}^{int} - \sum_{ij}^N \vec{r}_j \times \vec{F}_{ij}^{int}) = \frac{1}{2}\sum_{ij}^N (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ij}^{int} = \frac{1}{2}\sum_{ij}^N \vec{r}_{ij} \times \vec{F}_{ij}^{int}$$

If the force of interaction between the i^{th} and j^{th} particle is along the line joining the two particles, $\vec{F}_{ij}$ will be parallel to $\vec{r}_{ij}$. In that case, the above expression becomes zero. This is also known as strong law of action and reaction. Hence any interaction forces that obey strong law of action and reaction will be along the line joining the particles interacting.

Then the torque acting on such a system of particles and hence the rate of change of angular momentum will be

$$\vec{N} = \frac{d\vec{L}}{dt} = \sum_i^N \vec{r}_i \times \vec{F}_i^{ext}$$

Because the above torque depends only on external force, we shall call this as external torque. Hence, for a system of particles in which the interaction forces obey strong action and reaction law, the rate of change of angular momentum is equal to the total external torque acting on the system.

$$\frac{d\vec{L}}{dt} = \vec{N}^{ext}$$

If the total external torque acting on the system becomes zero, the total angular momentum of the system remains constant.

Thus the conservation of angular momentum of a system of particles can be stated as 'the total angular momentum of a system of particles gets conserved when the total external torque acting on the system becomes zero and the internal forces obey strong action and reaction law'.

2.3.3 Conservation of energy

In order to find the energy of the system, let us find the work done by all the forces - external as well as internal - in moving the system from initial configuration 1 to final configuration 2. The total work done in moving the system is equal to the sum of the work done in moving all the particles from configuration 1 to configuration 2. Thus,

$$W_{12} = \sum_i^N \int_1^2 \vec{F}_i \cdot d\vec{s}_i = \sum_i^N \int_1^2 \vec{F}_i \cdot d\vec{r}_i$$

$$W_{12} = \sum_i^N \int_1^2 \vec{F}_i^{ext} \cdot d\vec{r}_i + \sum_{ij}^N \int_1^2 \vec{F}_{ij}^{int} \cdot d\vec{r}_i$$

If the internal and the external forces are conservative, then they can be expressed in terms of corresponding potential energies. Thus, total force $\vec{F}_i$ on the i^{th} particle can be written as,

$$\vec{F}_i = \vec{F}_i^{ext} + \sum_j \vec{F}_{ij}^{int} = -\nabla_i V_i \tag{2.13}$$

Where the potential energy $V_i = V_i^{ext} + V_i^{int}$ is the sum of the potential energy functions of the external and internal forces. In the equation 2.13, the symbol ∇_i is

$$\nabla_i = \sum_i \hat{e}_i \frac{\partial}{\partial x_i}$$

and represents the gradient operator performing differentiation with respect to x_i, components of position vector $\vec{r}_i$ of the i^{th} particle. The operator can be written separately as

$$\vec{F}_i^{ext} = -\vec{\nabla}_i V_i^{ext} \qquad \text{and} \qquad \vec{F}_{ij}^{int} = -\vec{\nabla}_{ij} V_{ij}^{int}$$

Because the internal forces always exist in pairs, we can show that $V^{int} = \frac{1}{2}\sum_{ij} V_{ij}^{int}$. The factor $\frac{1}{2}$ occurs due to the fact that each term is being taken twice while summing over i and j.

The work done by the external forces will be

$$\sum_i \int_1^2 \vec{F}_i^{ext} \cdot d\vec{r}_i = -\sum_i \int_1^2 \vec{\nabla}_i V_i^{ext} \cdot d\vec{r}_i = \sum_i \int_1^2 dV_i^{ext} = -\sum_i V_i^{ext}\Big|_1^2 \tag{2.14}$$

$$\sum_i \int_1^2 \vec{F}_i^{ext} \cdot d\vec{r}_i = V_1^{ext} - V_2^{ext} \tag{2.15}$$

Similarly, the work done by the internal forces will be

$$\sum_{ij} \int_1^2 \vec{F}_{ij}^{int} \cdot d\vec{r}_i = \frac{1}{2}\sum_{ij} \int_1^2 \vec{F}_{ij}^{int} \cdot d\vec{r}_{ij} = -\frac{1}{2}\sum_{ij} \int_1^2 \vec{\nabla}_{ij} V_{ij}^{int} \cdot d\vec{r}_{ij}$$

$$\sum_{ij} \int_1^2 \vec{F}_{ij}^{int} \cdot d\vec{r}_i = -\frac{1}{2}\sum_{ij} \int_1^2 dV_{ij}^{int} = -V^{int}\Big|_1^2 = V_1^{int} - V_2^{int}$$

The total potential energy of the system is then given by

$$V = V^{ext} + V^{int}$$

In terms of the total potential energy of the system the work done is

$$W_{12} = -V\Big|_1^2 = V_1 - V_2 \tag{2.16}$$

The work done W_{12} can be expressed in terms of the difference between the kinetic energy of the system in the initial and final configurations as follows:

$$W_{12} = \sum_i \int_1^2 \vec{F}_i \cdot d\vec{r}_i = \sum_i \int_1^2 \frac{d}{dt}(m_i\dot{\vec{r}}_i) \cdot \dot{\vec{r}}_i dt = \sum m_i \int_1^2 \frac{d\dot{\vec{r}}_i}{dt} \cdot \dot{\vec{r}}_i \, dt = \sum m_i \int_1^2 \frac{d\vec{v}_i}{dt} \cdot \vec{v}_i \, dt$$

$$W_{12} = \sum m_i \int_1^2 \frac{1}{2}\frac{d(v_i^2)}{dt}\, dt = \int_1^2 d\big(\sum_i \frac{1}{2} m_i v_i^2\big) = T\Big|_1^2 = T_2 - T_1 \tag{2.17}$$

From the above two equations 2.16 and 2.17 we can conclude that

$$V_1 - V_2 = T_2 - T_1 \qquad \Longrightarrow \qquad T_1 + V_1 = T_2 + V_2$$

Thus the conservation of energy of the system of particles.

2.4 Constraints of motion

A motion that cannot proceed arbitrarily in any manner is called a constrained motion. This conditions which restrict the motion of the system are called constraints. For example, gas molecules in a container are constrained by the walls of the container. A particle placed on the surface of a sphere is restricted by the constraint so that it can only move on the surface or in the region exterior to the sphere. There are two main types of constraints: holonomic and non-holonomic.

2.4.1 Holonomic constraints

In holonomic constraints, the conditions of constraint are expressed as equations connecting the coordinates and time. If there are N number of particles in a system and there are k number of holonomic constraints, then there exist k number of independent equations connecting the position coordinates of N number of particles as

$$f_1(\vec{r}_1, \vec{r}_2, \vec{r}_3 ... \vec{r}_N, t) = 0$$

$$f_2(\vec{r}_1, \vec{r}_2, \vec{r}_3 ... \vec{r}_N, t) = 0$$

$$f_3(\vec{r}_1, \vec{r}_2, \vec{r}_3 ... \vec{r}_N, t) = 0$$

$$...$$

$$f_k(\vec{r}_1, \vec{r}_2, \vec{r}_3 ... \vec{r}_N, t) = 0$$

For example,

- In a rigid body, the distance between any two particles of the body remains constant. The equation of constraint will be

$$|\vec{r}_i - \vec{r}_j|^2 = c_{ij}^2$$

- The motion of an ant on the surface of a spherical surface. If x, y and z are the coordinates of the ant and a is the radius of the sphere, then the equation of constraint will be

$$x^2 + y^2 + z^2 = a^2$$

2.4.2 Non-holonomic constraints

Non-holonomic constraints are those which are not expressible in the form of an equation. The coordinates in this case are restricted either by inequalities or some differential equations. Sometimes, expressing non-holonomic constraints may not be possible using any mathematical expressions.

For example,

- The motion of a a gas molecule inside a spherical ball. If x, y and z are the coordinates of the molecule and a is the radius of the sphere, then the constraint is expressed as an inequality.

$$x^2 + y^2 + z^2 \leq a^2$$

- Motion of a vertical disc rolling on a horizontal plane is constrained by equations,

$$\dot{x} = v \sin\theta \qquad \text{and} \qquad \dot{y} = -v \cos\theta \qquad \text{with} \qquad v = a\dot{\phi}$$

  Where, θ is the angle made by the line joining the point of contact of the disc with the horizontal plane with x-axis parallel to y-axis. ϕ is the angle of rotation of the disc about its axis.

2.4.3 Scleronomous and Rheonomous constraints

Constraints are further classified as scleronomous and rheonomouns. A scleronomous constraint is one that is independent of time whereas a rheonomous constraint contains time as an explicit variable.

The motion of the bob attached to a pendulum with an inextensible string of length l is constrained by the equation

$$x^2 + y^2 = l^2$$

This is scleronomous constraint.

If the length of the pendulum is either increased or decreased as a function of time, then the constraint will turn out to be rheonomous.

$$x^2 + y^2 = lt^2$$

2.5 Generalized coordinates

The number of independent ways in which a mechanical system can move without violating any constraint is called number of degrees of freedom of the system. In other words, the degrees of freedom is the minimum number of variables required to describe the position/configuration of a system. If there are N number of particles in a system with no constraints, each particle would require three variables to specify their position, then the minimum number of variables required to specify the complete configuration of the system will be $3N$.

When there are constraints present in the motion of a system the motion becomes restricted. This intern creates an interdependency among various position coordinates. As a result of this the degrees of freedom of the system decreases. If there are K number of constraints present in the above example of a system of N number of particles, we will have K number of equations or inequalities that make the variables interdependent. Hence we can make K number of variables as dependent and the minimum number of variables required to describe the configuration of the system will become, $3N - K$.

But the interesting question is which among $3N$ variables must be selected as independent and which must be treated as dependent variables? Often we will have complete freedom in selecting and deciding the independent variables. In order to provide a general description, any $3N - k$ variables are chosen as independent variables and they are called generalized coordinates.

$$q_1,\ q_2,\ q_3\ ,\cdots\cdots, q_{3N-K}$$

In general these q_k's can be position coordinates, combination of position coordinates or any other variables associated with the system. That is the reason they are called generalized coordinates. The old position coordinates are expressed as functions of generalized coordinates and the generalized coordinates are often expressed as functions of position coordinates as

$$\vec{r}_i = \vec{r}_i(q_1, q_2, q_3, ...q_{3N-K}, t) \qquad \text{with } i = 1, 2, 3...N$$

$$q_k = q_k(\vec{r}_1, \vec{r}_2, \vec{r}_3...\vec{r}_N, t) \qquad \text{with } k = 1, 2, 3...3N - K$$

The time derivatives of these generalized coordinates are defined as generalized velocities: $\dot{q}_1, \dot{q}_2, \dot{q}_3, ...\dot{q}_{3N-K}$

2.5.1 Configuration space

To describe the motion of system of N number of particles having K number of constraints we require $3N - K$ number of generalized coordinates. Hence in a $3N - K$ dimensional space, the system can be represented by a point. This $3N - K$ dimensional space where the generalized coordinates are taken as axes is called configuration space. As different particles of the system moves the system traces a trajectory in configuration space. The path of the actual motion does not necessarily resemble the path in the configuration space.

2.6 Check your progress

Check your progress by answering the questions below.

1. When the linear and angular momenta of a particle gets conserved?
2. State the law of conservation of linear and angular momenta of a system of particles.
3. What is the necessary condition for energy to be conserved for a system of particles.
4. What are constraints of motion?
5. What are generalized coordinates.

2.7 Keywords

- Conservation of linear momentum
- Conservation of angular momentum
- Conservation of energy
- Strong action and reaction law
- Constraints
- Generalized coordinates

2.8 Worked examples

1. Three particles of masses 2kg, 2.5kg and 3.5kg are placed at points (2,0,2), (0,3,2) and (-2,-2,0) respectively. Determine the coordinates of the center of mass.

   Answer:

   Data:

   $$m_1 = 2kg \quad r_1 = (2,0,2)$$
   $$m_2 = 2.5kg \quad r_2 = (0,3,2)$$
   $$m_3 = 3.5kg \quad r_3 = (-2,-2,0)$$

   We know the expression for the position vector of center of mass is

   $$R = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i}$$

   $$R = \frac{m_1(x_1,y_1,z_1) + m_2(x_2,y_2,z_2) + m_3(x_3,y_3,z_3)}{m_1+m_2+m_3}$$

   $$R = \frac{2(2,0,2) + 2.5(0,3,2) + 3.5(-2,-2,0)}{2+2.5+3.5} = \frac{4-7, 7.5-7, 4+5}{8} = (-0.375, 0.0625, 1.125)$$

   The coordinates of center of mass is (-0.375, 0.0625, 1.125)

2. Show that the number of generalized coordinates required to describe a rigid body is 6.

   Answer:

   Data:

   A rigid body is defined as a system of particles in which the distance between any two particle is fixed.

   The number of generalized coordinates required to describe a system is equal to its degrees of freedom. To find the degrees of freedom of a rigid body, let us start with a single particle.

   The degrees of freedom of a single particle is 3 as it can move in any combination of three mutually perpendicular directions in three dimensional space.

If there are two such particles, the total degrees of freedom will be 6 (three for each particle). Let us introduce a constraint saying the distance between the two particle is fixed. Then number of constraint is 1

Then the degree of freedom of this two particle system will be $3N - k = 3 \times 2 - 1 = 5$.

Now let us introduce another particle with a constraint that the distance of the new particle from the previous two particles are fixed. Hence the total number of particles is 3 and total number of constraints is 3. Then the degree of freedom would be $3N - k = 3 \times 3 - 3 = 6$.

Let us introduce the fourth particle with constraints that its distance from the other three particles are fixed. Hence the total number of constraints will be 6. Then the degrees of freedom would be $3N - k = 3 \times 4 - 6 = 6$

For any new particle introduced, the number of constraints also increase proportionately that keeps the degrees of freedom of rigid body at 6. Hence the number of generalized coordinates required to describe the motion of the rigid body is 6.

3. A particle of mass $200g$ is moving with an initial velocity of 15 ms^{-1} along x-axis. If a force of 2 N is applied on to the particle for 2.5 seconds along y-axis, determine the final velocity and kinetic energy of the particle.

   Answer: The particle is initially moving along x direction with a velocity of 15 ms^{-1}. It will continue to move with that velocity unless a force parallel to x-axis is not applied.

   $v_x = 15ms^{-1}$

   The force of 2 N is being applied along y-axis produces an acceleration of $a = F/m = 2/0.2 = 10ms^{-2}$. Hence after 4 seconds, the velocity along y-axis will be,
   $v_y = \frac{1}{2}at^2 = \frac{1}{2} \times 10 \times 2.5^2 = 31.25ms^{-1}$
   Then the final velocity will be $\vec{v} = 15\hat{i} + 31.25\hat{j}$

   The kinetic energy will be

   $$T = \frac{1}{2}m(v_x^2 + v_y^2) = \frac{1}{2} \times 0.2 \times (15^2 + 31.25^2) = 120.15J$$

4. Check whether the force field $\vec{F} = (x^2 - xy)\hat{i} + (y^2 - xy)\hat{j}$ is conservative or not.

   Answer: For a conservative force field the force can be obtained as negative gradient of potential. If V is the potential, then

$$\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath} = -\frac{\partial V}{\partial x}\hat{\imath} - \frac{\partial V}{\partial y}\hat{\jmath}$$

Then we can write

$$\frac{\partial V}{\partial x} = xy - x^2 \qquad \text{and} \qquad \frac{\partial V}{\partial x} = xy - y^2$$

Now if we integrate the above equations and if we can get the same potential function we can conclude that is the potential function to derive the above force and hence the force field is conservative in nature.

$$V_x = \int (xy - x^2)dx \qquad \text{and} \qquad V_y = \int (xy - y^2)dy$$

$$V_x = \frac{x^2y}{2} - \frac{x^3}{3} + f(y) \qquad \text{and} \qquad V_y = \frac{xy^2}{2} - \frac{y^3}{3} + f(x)$$

We can not combine the above two to obtain one common function. Hence the given force field is not conservative.

2.9 Questions for self study

1. State prove the law of conservation of linear momentum of a single particle.
2. State and prove the law of conservation of angular momentum of a single particle.
3. State and prove the law of conservation of energy of a single particle.
4. State prove the law of conservation of linear momentum of a system of particles.
5. State and prove the law of conservation of angular momentum of a system of particles.
6. State and prove the law of conservation of energy of a system of particles.
7. What are constraints? Explain the various types of constraints.
8. What are generalized coordinates? Explain their significance.

2.10 Answers to check your progress

1. When the total external force acting on a particle becomes zero, its linear momentum gets conserved. When the total external torque acting on a particle becomes zero, the angular momentum of a particle gets conserved.

2. When the total external force acting on the system of particle becomes zero, the linear momentum of the system of particle gets conserved. When the total external torque acting on the system becomes zero and when the internal forces obey strong action and reaction law, the angular momentum of the system gets conserved.

3. The energy of a system of particles gets conserved when both internal and external forces are conservative in nature.

4. Conditions imposed on motion of a system are known as constraints.

5. Generalized coordinates are variables that are equal to degrees of freedom in number used to describe the configuration of any physical system.

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UNIT-3: Lagrangian Formulation

D' Alembert's principle and Lagrange's equations of motion, Hamilton's principle of least action, Lagrange's equations from Hamilton's principle, velocity dependent potentials.

3.0 Objectives

After studying this unit you will be able to

- Describe the principle of virtual displacement and virtual work.
- Obtain the equations of motion from D'Alembert's principle of virtual work.
- Describe the Hamilton's principle of least action.
- Deduce the Lagrange's equations of motion using Hamilton's principle of least action.
- Describe the velocity dependent potential.

3.1 Introduction

Newton's laws of motion provide a deterministic approach to find the equations of motion of physical system. This means when all the forces acting on a physical system are specified and when the initial conditions are known, the equations of motion that describe the position/configuration of the system can be uniquely determined. This is evident, but may not be easy at all the time. Knowing the initial conditions is not a problem, but specifying all the forces acting on the system may not be possible at all the time. The forces due to external force fields may be easy to determine, but the forces due to constraints are often difficult to determine. For example, when a circular disc rolls on a horizontal surface, it experiences a constraint force due to the normal reaction from the surface. When a molecule is bound inside a container, the walls of the container exert force on the molecules whenever they tend to go away from the container. These forces are often difficult to specify. On the other hand, these constraint forces do not directly contribute to the motion, but without specifying them the equations of motion can not be determined in Newtonian mechanics.

Analytical mechanics that includes several different formalism provide alternative approaches to Newtonian mechanics. Lagrangian formulation is one of them that is highly effective an useful to find the equations of motion of physical systems when the forces of constraints are unknown. The formalism is so defined that the constraint forces are not necessary to find the equations of motion of the system.

3.2 D′ Alembert's principle of virtual work

Consider a system described by n generalized coordinates $q_j(j = 1, 2, 3, ...n)$. Suppose the system undergoes a certain displacement in the configuration space in such a way that it does not take any time and that it is consistent with the constraints on the system. Such displacements are called virtual displacement because they do not represent actual motion of the system, the work doen by the forces of constraint in such a virtual displacement is zero. If this case, the virtual displacement is taken at right angles to the direction of the force i.e., along the surface, so that the work done by the force during the virtual displacement is zero.

Let the virtual displacement of the i^{th} particle of the given system be $\delta\vec{r}_i$. If the given system is in equilibrium, the resultant force acting on the i^{th} particle of the system must be zero, i.e. $\vec{F}_i = 0$. It is the obvious that virtual work $\vec{F}_i \cdot \delta\vec{r}_i = 0$ for the i^{th} particle and hence it is also zero for all the particles of the system. Thus

$$dW = \sum_i \vec{F}_i \cdot \delta\vec{r}_i = 0$$

The resultant force $\vec{F}_i$ acting on the i^{th} particle is made up of the two forces: $\vec{F}^a$, the applied force and $\vec{f}_i$, the force of constraint. Hence we can write,

$$\vec{F}_i = \vec{F}_i^a + \vec{f}_i$$

Then the corresponding work done under virtual displacement would be

$$\sum_i \vec{F}_i^a \cdot \delta\vec{r}_i + \sum_i \vec{f}_i \cdot \delta\vec{r}_i = \sum_i \vec{F}_i \cdot \delta\vec{r}_i = 0$$

Because the virtual displacement of the system is consistent with the forces of constraints, the work done due to constraint forces on virtual displacement becomes zero.

$$\sum_i \vec{f}_i \cdot \delta\vec{r}_i = 0$$

With this restriction we arrive at the principle of virtual work which states that the virtual work done by the applied forces acting on a system in equilibrium is zero. Thus we have

$$\sum_i \vec{F}_i \cdot \delta\vec{r}_i = 0$$

This is also known as D'Alembert's principle of virtual work.

It should be noted that all the coordinates and hence the virtual displacements $\delta\vec{r}_i$ are not independent of each other. In fact, some of these must be connected by the equations of constraints. Hence we cannot treat virtual displacements $\delta\vec{r}_i$ as completely arbitrary and equate their coefficients. For this purpose, we shall have to transform coordinates $\vec{r}_i$ to independent generalized coordinates q_j. There is another point that needs consideration. Most of the systems we come across in mechanics are not in static equilibrium. hence the principle must be modified to include the dynamic systems as well. This can be achieved by considering the Newton's equation of motion as,

$$\vec{F}_i = \dot{\vec{p}}_i \qquad \Longrightarrow \qquad \vec{F}_i - \dot{\vec{p}}_i = 0$$

$\Longrightarrow$

$$\sum(\vec{F}_i - \dot{\vec{p}}_i) \cdot \delta\vec{r}_i = 0$$

3.3 Lagrange's equations of motion from D' Alembert's principle

In order to obtain a constraint independent equations of motion, we have to transfer the D'Alembrt's principle of virtual work in terms of generalized coordinates. We know that position coordinates can be expressed as functions of generalized coordinates as

$$\vec{r}_i = \vec{r}_i(q_j) \qquad i = 1, 2, 3...N \text{ and } j = 1, 2, 3...n = 3N - k$$

consider an infinitesimal virtual displacement $\delta\vec{r}_i$ at a particular instant of time t.

$$\delta\vec{r}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j}\delta q_j$$

The variation with respect to the time is absent in the above equation because the virtual displacement is assumed to take place at fixed instant of time. Further, the velocities are given by

$$\vec{v}_i = \dot{\vec{r}}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \vec{r}_i}{\partial t}$$

In the above equation, $\dot{q}_j = \dfrac{\partial q_j}{\partial t}$ are the generalized velocities. It should be noted that since generalized coordinates q_j need not have the dimensions of length, generalized velocities need not have the dimensions of velocity.

The virtual work done by forces $\vec{F}_i$ in terms of virtual displacements $\delta\vec{r}_i$ is given by

$$\delta W = \sum_i \vec{F}_i \cdot \delta\vec{r}_i$$

$$\delta W = \sum_j \left(\sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right)\delta q_j$$

$$\sum_i \vec{F}_i \cdot \delta\vec{r}_i = \sum_j Q_j \delta q_j \tag{3.1}$$

Where $Q_j = \sum_i \vec{F}_i \cdot \dfrac{\partial \vec{r}_i}{\partial q_j}$ represents the j^{th} component of the generalized force.

Consider the second term in the modified D'Alembert's principle of virtual work

$$\sum_i \dot{\vec{p}}_i \cdot \delta\vec{r}_i = \sum_i \frac{d}{dt}(m_i \dot{\vec{r}}_i) \cdot \delta\vec{r}_i = \sum_{i,j} m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\delta q_j$$

The coefficient of δq_j on the right hand side of the above equation can be written as

$$\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i \left[\frac{d}{dt}\left(m_i \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right) - m_i \dot{\vec{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)\right] \tag{3.2}$$

In the above expression,

$$\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right) = \sum_k \frac{\partial^2 \vec{r}_i}{\partial q_j \partial q_k}\dot{q}_k + \frac{\partial^2 \vec{r}_i}{\partial q_j \partial t} = \frac{\partial \dot{\vec{r}}_i}{\partial q_j} \tag{3.3}$$

Similarly we can also show that

$$\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial q_j} \tag{3.4}$$

Using expression in equations 3.3 and 3.4 in equation 3.2, we get,

$$\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i \Big[\frac{d}{dt}(m_i \dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j}) - m_i \dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial q_j}\Big] \tag{3.5}$$

$$= \frac{d}{dt}\Big[\frac{\partial}{\partial \dot{q}_j}(\sum_i \frac{1}{2} m_i |\dot{\vec{r}}_i|^2)\Big] - \Big[\frac{\partial}{\partial q_j}(\sum_i \frac{1}{2} m_i |\dot{\vec{r}}_i|^2)\Big] \tag{3.6}$$

The term $\big(\sum_i \frac{1}{2} m_i |\dot{\vec{r}}_i|^2\big)$ is nothing but the total kinetic energy of the system of particles. Let this kinetic energy be denoted by T. Then we have

$$\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \frac{d}{dt}\Big(\frac{\partial T}{\partial \dot{q}_j}\Big) - \frac{\partial T}{\partial q_j} \tag{3.7}$$

Now combining the equations 3.1 and 3.7 we can rewrite the D'Alembert's principle of virtual work as

$$\sum_j \Big[\Big(\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_j}) - \frac{\partial T}{\partial q_j}\Big) - Q_j\Big]\delta q_j = 0 \tag{3.8}$$

Because each generalized coordinate is independent of other generalized coordinates, we can equate each term of the above summation to zero. Thus we can write,

$$\Big(\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_j}) - \frac{\partial T}{\partial q_j}\Big) - Q_j = 0 \tag{3.9}$$

The above equation is actually a collection of n number of equations which together are called Lagrange's equations of motion.

If the forces acting on the particles are conservative in nature, they can be expressed as negative gradient of a scalar potential as

$$\vec{F}_i = -\vec{\nabla}_i V$$

The generalized force will be

$$Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i -\vec{\nabla}_i V \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

Using the above expression for the generalized force in the Lagrange's equation we can write,

$$\frac{d}{dt}\Big(\frac{\partial T}{\partial \dot{q}_j}\Big) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = 0 \tag{3.10}$$

$$\frac{d}{dt}\Big(\frac{\partial T}{\partial \dot{q}_j}\Big) - \frac{\partial (T-V)}{\partial q_j} = 0 \tag{3.11}$$

If we assume that the potential energy is just a function of position not velocity. Then the derivative of the potential with respect to the velocity, will be zero. This is true for generalized velocities also. Then addition or subtraction of $\frac{\partial V}{\partial \dot{q}_j}$ in the above equation will not contribute in any manner. Hence we shall include $\frac{\partial V}{\partial \dot{q}_j}$ in the first term of the above equation to attain a uniformity. Note that this can not be done for velocity dependent potentials. Then we get

$$\frac{d}{dt}\Big(\frac{\partial (T-V)}{\partial \dot{q}_j}\Big) - \frac{\partial (T-V)}{\partial q_j} = 0 \tag{3.12}$$

In the above equation we shall define a new function called Lagrangian of the system as the difference between the kinetic and potential energy function as, $L = T - V$. Then the Lagrange's equations of motion in terms of the Lagrangian of the system will become

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_j}\Big) - \frac{\partial L}{\partial q_j} = 0 \tag{3.13}$$

These equations are also known as Euler-Lagrange's equations of motion.

3.4 Hamilton's principle of least action

Hamilton's principle of least action is stated as follows 'Of all the possible paths along which a dynamical system may move from one point to another within a given interval of time consistent with constraints, the actual path followed is that which minimizes the time intgral of the Lagrangian'.

The principle can alternatively stated as:

The motion of a system from an instant t_1 to another instant t_2 is such that the line integral $\int_{t_1}^{t_2} L dt$ with $L = T - V$ is an extremum for the actual path of the motion. In terms of calculus of variation, we can state the Hamilton's principle as

$$\delta J = \delta \int_{t_1}^{t_2} L dt = 0$$

3.5 Lagrange's equations of motion from Hamilton's principle

The fundamental problem of the calculus of variations is easily generalized to the case where f is a function of many independent variables y_i, and their derivatives $\dot{y}_i$. Then the variation of the integral J

$$\delta J = \delta \int_1^2 f(y_1(x); y_2(x); ..., \dot{y}_1(x); \dot{y}_2(x)..., x) dx$$

is obtained by considering J as a function of parameter α that labels a possible set curves $y_1(x, \alpha)$. Thuswe may introduce α by setting

$$y_i(x, \alpha) = y_i(x, 0) + \alpha \eta_i(x)$$

The variation of Jcan be taken as

$$\frac{\partial J}{\partial \alpha} d\alpha = \int_1^2 \sum_i \Big(\frac{\partial f}{\partial y_i} \frac{\partial y_i}{\partial \alpha} d\alpha + \frac{\partial f}{\partial \dot{y}_i} \frac{\partial \dot{y}_i}{\partial \alpha} d\alpha \Big) dx$$

Again we integrate by parts the integral involved in the second sum of the above equation

$$\int_1^2 \frac{\delta f}{\partial \dot{y}_i} \frac{\partial^2 y_i}{\partial \alpha} dx = \frac{\partial f}{\partial \dot{y}_i} \frac{\partial y_i}{\partial \alpha} \Big|_1^2 - \int_1^2 \frac{\partial y_i}{\partial \alpha} \frac{d}{dx} \Big(\frac{\partial f}{\partial \dot{y}} \Big) dx$$

The first term of the above expression vanishes because all curves pass through the fixed end points. The the variation in the integral J becomes,

$$\delta J = \int_1^2 \sum_i \Big[\frac{\partial f}{\partial y_i} - \frac{d}{dx} \Big(\frac{\partial f}{\partial \dot{y}_i} \Big) \Big] \delta y_i dx \tag{3.14}$$

Because each y_i are independent, the coefficients of each δy_i vanish in above equation. Hence,

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx} \Big(\frac{\partial f}{\partial \dot{y}_i} \Big) \tag{3.15}$$

According to Hamilton's principle of least action, the integral

$$I = \int_1^2 L(q_i, \dot{q}_i, t)dt$$

Then the corresponding equations of motion can be obtained from the above integral function by replacing f by L, y_i by q_i.

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_i}\Big) = 0 \tag{3.16}$$

These are nothing but Euler Lagrange equations of motion.

3.6 Velocity dependent potentials

When the potentials depend on the velocity, the expression for the generalized force will be

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\Big(\frac{\partial U}{\partial \dot{q}_j}\Big)$$

In the above expression, $U(q_j, \dot{q}_j)$ is called generalized potential or the velocity dependednt potential.

Then the Lagrangian of the system can be taken as

$$L = T - U$$

And the Lagrangian equations of motion will be

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_i}\Big) = 0$$

3.6.1 Velocity dependent potential of electromagnetic fields

Whenever the force acting on a particle of system of particles becomes a function of velocity, the corresponding potentials used to derive the force also become velocity dependent potentials. We come across velocity dependent potential when we consider the electromagnetic forces acting on moving charges.

Consider an electric charge, q of mass m moving at a velocity $\vec{v}$, in an region where both electric field of intensity $\vec{E}$ and magnetic field of intensity $\vec{B}$ are present. The force experienced by the charge is also known as Lorentz force is given by

$$F = q[\vec{E} + (\vec{v} \times \vec{B})]$$

Both $\vec{E}(x, y, z, t)$ and $\vec{B}(x, y, z, t)$ are continuous functions of space and time and are derivable from a scalar potential $\phi(x, y, z, t)$ and a vector potential $\vec{A}(x, y, z, t)$ as

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \qquad \text{and} \qquad \vec{B} = \vec{\nabla} \times \vec{A}$$

The force on the charge can be derived from the following velocity-dependent potential energy

$$U = q\phi - q\vec{A} \cdot \vec{v}$$

Then the Lagrangian of the charged particle will be

$$L = T - U = \frac{1}{2}mv^2 - q\phi + q\vec{A} \cdot \vec{v}$$

We can find the equations of motion of the charged particle using the above Lagrangian when the definite functional form of the scalar and vector potentials are known. With this general description also we can compute the equations of motion that will conclude in the expression for the Lorentz force.

3.7 Check your progress

Check your progress by answering the questions below.

1. What is virtual displacement?
2. State D'Alembert's principle of virtual work.
3. State the Hamilton's principle of least action.
4. What are velocity dependent potentials?

3.8 Keywords

- Virtual displacement

- Virtual work
- Lagrangian
- Equations of motion
- Velocity dependent potential

3.9 Worked examples

1. Consider a particle moving in space. Using the spherical polar coordinates (r, θ, ϕ) as the generalized coordinates, express the virtual displacements δx, δy and δz in terms of r, θ and ϕ

   Answer:

   In terms of spherical polar coordinates (r, θ, ϕ), we have the expressions for x, y and z as

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$$

$$\frac{\partial x}{\partial r} = \sin\theta\cos\phi \qquad \frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi \qquad \frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi$$

$$\frac{\partial y}{\partial r} = \sin\theta\sin\phi \qquad \frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi \qquad \frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi$$

$$\frac{\partial z}{\partial r} = \cos\theta \qquad \frac{\partial z}{\partial \theta} = -r\sin\theta \qquad \frac{\partial z}{\partial \phi} = 0$$

   The virtual displacement about x, y and z will be

$$\delta x = \frac{\partial x}{\partial r}\delta r + \frac{\partial x}{\partial \theta}\delta\theta + \frac{\partial x}{\partial \phi}\delta\phi$$

$$\delta x = \sin\theta\cos\phi\delta r + r\cos\theta\cos\phi\delta\theta - r\sin\theta\sin\phi\delta\phi$$

   Similarly,

$$\delta y = \frac{\partial y}{\partial r}\delta r + \frac{\partial y}{\partial \theta}\delta\theta + \frac{\partial y}{\partial \phi}\delta\phi$$

$$\delta x = \sin\theta\sin\phi\delta r + r\cos\theta\sin\phi\delta\theta + r\sin\theta\cos\phi\delta\phi$$

$$\delta z = \frac{\partial z}{\partial r}\delta r + \frac{\partial z}{\partial \theta}\delta\theta + \frac{\partial z}{\partial \phi}\delta\phi$$

$$\delta x = \cos\theta\delta r + -r\sin\theta\delta\theta$$

2. Consider the motion of a particle of mass m moving in space. Selecting the cylindrical coordinates (ρ, ϕ, z) as the generalized coordinates, calculate the generalized force components acting if a force $\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$ acts on it.

   Answer:

   We know that the generalized force is defined as

$$Q_j = \sum_i F_i \frac{\partial x_i}{\partial q_j} = F_x \frac{\partial x}{\partial q_j} + F_y \frac{\partial y}{\partial q_j} + F_z \frac{\partial z}{\partial q_j}$$

   Then,

$$Q_\rho = F_x \frac{\partial x}{\partial \rho} + F_y \frac{\partial y}{\partial \rho} + F_z \frac{\partial z}{\partial \rho}$$

$$Q_\phi = F_x \frac{\partial x}{\partial \phi} + F_y \frac{\partial y}{\partial \phi} + F_z \frac{\partial z}{\partial \phi}$$

$$Q_j = F_x \frac{\partial x}{\partial z} + F_y \frac{\partial y}{\partial z} + F_z \frac{\partial z}{\partial z}$$

   To find the generalized force, let us find the transformation partial derivatives,

$$x = \rho\cos\phi \qquad y = \rho\sin\phi \qquad z = z$$

   Therefore,

$$\frac{\partial x}{\partial \rho} = \cos\phi \qquad \frac{\partial y}{\partial \rho} = \sin\phi \qquad \frac{\partial z}{\partial \rho} = 0$$

$$\frac{\partial x}{\partial \phi} = -\rho\sin\phi \qquad \frac{\partial y}{\partial \phi} = \rho\cos\phi \qquad \frac{\partial z}{\partial \phi} = 0$$

$$\frac{\partial x}{\partial z} = 0 \qquad \frac{\partial y}{\partial z} = 0 \qquad \frac{\partial z}{\partial z} = 1$$

   Then the components of generalized force will be

$$Q_\rho = F_x\cos\phi + F_y\sin\phi = F_\rho$$

$$Q_\phi = -F_x\rho\sin\phi + F_y\rho\cos\phi = \rho F_\phi$$

$$Q_z = F_z$$

3. Two equal masses m connected by a massless rigid rod of length l forming a dumb-bell is rotated in the xy plane. Find the Lagrangian of the system.

Answer:

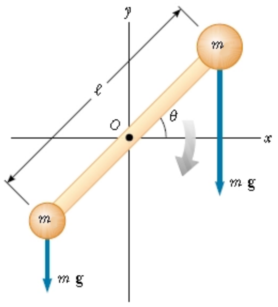

Figure 3.7: A dumbbell in xy-plane

The system has three degrees of freedom. The cartesian coordinates x_1, y_1 and the θ can be selected as the generalized coordinates. From the figure we can write,

$$x_2 = x_1 + l\cos\theta \qquad \text{and} \qquad y_2 = y_1 + l\sin\theta$$

$\Longrightarrow$

$$\dot{x}_2 = \dot{x}_1 - l\sin\theta\dot{\theta} \qquad \text{and} \qquad \dot{y}_2 = \dot{y}_1 + l\cos\theta\dot{\theta}$$

The kinetic energy of the system will be

$$T = \frac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m(\dot{x}_2^2 + \dot{y}_2^2)$$

substituting the expression for $\dot{x}_2$ and $\dot{y}_2$ in the above expression,

$$T = m(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m(l^2\dot{\theta}^2 - 1l\dot{x}_1\dot{\theta}\sin\theta + 2l\dot{y}_1\dot{\theta}\cos\theta)$$

The potential energy of the system is given by

$$V = mgy_1 + mgy_2 = 2mgy_1 + mgl\sin\theta$$

Then the Lagrangian of the system will be

$$L = m(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m(l^2\dot{\theta}^2 - 1l\dot{x}_1\dot{\theta}\sin\theta + 2l\dot{y}_1\dot{\theta}\cos\theta) - 2mgy_1 + mgl\sin\theta$$

We can use the above Lagrangian to find the equations of motion. If we substitute the derivative of the above Lagrangian in the Lagrangian equations of motion we get the equations of motion as,

$$2\ddot{x}_1 - l\ddot{\theta}\sin\theta - l\dot{\theta}^2\cos\theta = 0$$

$$2\ddot{y}_1 + l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta = 0$$

$$l^2\ddot{\theta} - l\ddot{x}_1\sin\theta + l\ddot{y}_1\cos\theta + gl\cos\theta = 0$$

3.10 Questions for self study

1. What is virtual displacement? State and Explain D'Alembert's principle of virtual work.
2. Derive the Lagrangian equations of motion from the D'Alembert's principle of virtual work.
3. State and explain Hamilton's principle of least action.
4. Deduce the Lagrangian equations of motion from Hamilton's principle of least action.
5. Describe electromagnetic fields as velocity dependent potential.

3.11 Answers to check your progress

1. Displacement of the configuration of the system at an instant of time being consistent with forces of constraints is known as virtual displacement.
2. D'Alembert's principle of virtual work states as 'the virtual work done by the external forces acting on a system in equilibrium is zero'.
3. Hamilton's principle of least action states that 'Of all the possible paths along which a dynamical system may move from one point to another within a given interval of time consistent with constraints, the actual path followed is that which minimizes the time integral of the Lagrangian'.
4. Potentials that depend on velocities are called velocity dependent potentials. These arise when the forces depend on velocity.

3.12 References

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UNIT-4: Applications of Lagrangian formulation

Computing Lagrange's equations, conservation theorems and symmetry properties, determination of equations of motion for some example problems.

4.0 Objectives

After studying this unit you will be able to

- Describe the method to construct the Lagrangian and use it to find the equations of motion of a system.
- Use the Lagrangian mechanics to find the equations of motion of several simple mechanical systems.
- Describe the conservation theorems and symmetry in Lagrangian.

4.1 Introduction

We learned about the Lagrangian formulation in the previous unit. In this unit we shall learn how to use the Lagrangian formulation to determine the equations of motion of some example systems.

4.2 Constructing Lagrangian

The Lagrangian of a system is defined as a function that is difference between the kinetic and potential energy functions.

$$L = T - V$$

In order to determine the equations of motion using Lagrangian approach the following steps must be followed.

1. First we have to decide the generalized coordinates of the system. Note that they are not unique, they are considered based on the mathematical convenience.

2. Once the generalized coordinates are decided, kinetic energy and potential energies are expressed in terms of those generalized coordinates and corresponding generalized velocities.

3. The Lagrangian is constructed by taking the difference between the kinetic and potential energy functions.

4. The derivatives of Lagrangian is determined with respect the the generalized coordinates and generalized velocities.

5. Then the Lagrangian equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

is used to determine the equations of motion.

4.3 Determination of equations of motion using Lagrangian formulation

Now let us use the above described method to find the equations of motion of some example systems.

4.3.1 Free particle in Cartesian coordinate system

Consider a free particle of mass m moving in a three dimensional space. The free particle means it is not experiencing any king of force. Hence its potential energy will also be zero. Thus the Lagrangian of the particle is equal to the kinetic energy of the particle.

Let us use the Cartesian coordinates x, y and z as the generalized coordinates. Hence $\frac{dx}{dt} = \dot{x}$, $\frac{dy}{dt} = \dot{y}$ and $\frac{dz}{dt} = \dot{z}$ will be the generalized velocities.

Then the expression for kinetic energy will be

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left[\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right]$$

Because the potential energy of the free particle is zero, the Lagrangian can be taken as

$$L = T - V = \frac{1}{2}m\left[\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right] - 0$$

$$L = \frac{1}{2}m\left[\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right]$$

Let us find the derivatives of the Lagrangian with respect to the generalized coordinates and generalized velocities. Note that there are no generalized coordinates present in the expression for Lagrangian. Hence all the derivatives of Lagrangian with respect to the coordinates becomes zero.

$$\frac{\partial L}{\partial x} = 0, \qquad \frac{\partial L}{\partial y} = 0, \qquad \frac{\partial L}{\partial z} = 0$$

The derivatives of Lagrangian with respect to the velocities will be

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \qquad \frac{\partial L}{\partial \dot{y}} = m\dot{y}, \qquad \frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

We know the general equation of motion in Lagrangian formulation,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad \text{with } q_1 = x,\ \ q_2 = y,\ \ q_3 = z$$

Using the derivatives of the Lagrangian with respect the the generalized coordinates and velocities in the above expression we get,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x} = 0 \qquad \Longrightarrow \qquad \frac{d}{dt}\left(m\dot{x}\right) - 0 = 0 \qquad \Longrightarrow \qquad m\ddot{x} = 0$$

Similarly,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_i}\right) - \frac{\partial L}{\partial y} = 0 \qquad \Longrightarrow \qquad \frac{d}{dt}\left(m\dot{y}\right) - 0 = 0 \qquad \Longrightarrow \qquad m\ddot{y} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}_i}\right) - \frac{\partial L}{\partial z} = 0 \qquad \Longrightarrow \qquad \frac{d}{dt}\left(m\dot{z}\right) - 0 = 0 \qquad \Longrightarrow \qquad m\ddot{z} = 0$$

Hence, the equations of motion of the particle will be

$$m\ddot{x} = 0; \qquad m\ddot{y} = 0; \qquad m\ddot{z} = 0$$

On integration, the above set of equations will give,

$$m\dot{x} = C_{1x}; \qquad m\dot{y} = C_{1y}; \qquad m\dot{z} = C_{1z}$$

Further integration will give us,

$$mx = C_{1x}t + C_{2x}; \qquad my = C_{1y}t + C_{2y}; \qquad mz = C_{1z}t + C_{2z}$$

In the above expressions, C's are constants of integration which can be determined with known initial conditions. Note that the above equations are equations of straight line. Hence, we can conclude that the free particle travel along straight line.

If the particle was at origin with coordinates $(0,0,0)$ with a initial velocity u_x, u_y and u_z along x, y and z axes respectively at $t = 0$, the above general equations of motion reduces to

$$x = u_x t; \qquad y = u_y t; \qquad z = u_z t$$

Note: Similar to generalized coordinates and generalized velocities, we also define generalized momenta. However this is not defined as product of mass and generalized velocity but as the derivative of the Lagrangian with respect to the generalized velocities.

$$p_j = \frac{\partial L}{\partial \dot{q}_i}$$

In the above problem, the generalized momenta are same as the linear momenta along x, y and z axes because we are using Cartesian coordinate system. Further, there are some interesting facts regarding this generalized momenta that we shall discuss after discussing few more examples.

4.3.2 Free particle in spherical polar coordinate system

Once again consider a free particle of mass m moving in a three dimensional space. Let us use the spherical polar coordinates r, θ and ϕ as the generalized coordinates. Hence $\frac{dr}{dt} = \dot{r}$, $\frac{d\theta}{dt} = \dot{\theta}$ and $\frac{d\phi}{dt} = \dot{\phi}$ will be the generalized velocities.

The three dimensional velocity of the particle in spherical polar coordinate system can be taken as

$$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\hat{\phi}$$

Then the kinetic energy of the particle would be

$$T = \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right]$$

Because we have considered a free particle, the potential energy is zero. Hence the Lagrangian will be equal to the kinetic energy itself. Thus we can take the Lagrangian as

$$L = T - V = \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right]$$

Now let us find the derivatives of the Lagrangian with respect to the generalized coordinates and generalized velocities

$$\frac{\partial L}{\partial r} = \frac{1}{2}m\left[0 + 2r\dot{\theta}^2 + 2r\sin^2\theta\dot{\phi}^2\right] = mr(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)$$

$$\frac{\partial L}{\partial \theta} = \frac{1}{2}m\left[0 + 0 + r^2 2\sin\theta\cos\theta\dot{\phi}^2\right] = mr^2\sin 2\theta\dot{\phi}^2$$

$$\frac{\partial L}{\partial \phi} = \frac{1}{2}m[0 + 0 + 0] = 0$$

$$\frac{\partial L}{\partial \dot{r}} = \frac{1}{2}m\left[2\dot{r} + 0 + 0\right] = m\dot{r}$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}m\left[0 + r^2 2\dot{\theta} + 0\right] = mr^2\dot{\theta}$$

$$\frac{\partial L}{\partial \dot{\phi}} = \frac{1}{2}m[0 + 0 + r^2\sin^2\theta 2\dot{\phi}] = mr^2\sin^2\theta\dot{\phi}$$

Now let us use the above derivatives in Lagrangian equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad \text{with } q_1 = r,\ \ q_2 = \theta,\ \ q_3 = \phi$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \qquad \Longrightarrow \qquad \frac{d}{dt}(m\dot{r}) - mr(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \qquad \Longrightarrow \qquad \frac{d}{dt}(mr^2\dot{\theta}) - mr^2\sin 2\theta\dot{\phi}^2 = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0 \qquad \Longrightarrow \qquad \frac{d}{dt}(mr^2 \sin^2\theta\dot{\phi}) - 0 = 0$$

Thus, we have the three equations of motion of the free particle in spherical polar coordinate system as

$$\frac{d}{dt}(m\dot{r}) = mr(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) \qquad \frac{d}{dt}(mr^2\dot{\theta}) = mr^2 \sin 2\theta\dot{\phi}^2 \qquad \frac{d}{dt}(mr^2 \sin^2\theta\dot{\phi}) = 0$$

The third equation implies $mr^2 \sin^2\theta\dot{\phi} = constant$. Let that constant be denoted by l_ϕ and it represents the generalized momentum with respect to ϕ. We also call this as azimuthal angular momentum. Hence we can observed that for a free particle, the azimuthal angular momentum is conserved. From this equation we can write,

$$\dot{\phi} = \frac{l_\phi}{mr^2 \sin^2\theta}$$

Using this in the second equation of motion, we get,

$$\frac{d}{dt}(mr^2\dot{\theta}) = mr^2 \sin 2\theta\dot{\phi}^2 = mr^2 \sin 2\theta \frac{l_\phi^2}{m^2 r^4 \sin^4\theta} = \frac{l_\phi^2 \sin 2\theta}{mr^2 \sin^4\theta}$$

4.3.3 Simple harmonic oscillator

Consider a simple harmonic oscillator constructed using a mass attached to a spring as shown in the figure. Let m be the mass and k be the spring constant. Let the mean position coincides with the origin and x be the position of the mass at any instant and is the generalized coordinate. Then $\dot{x}$ will be the generalized velocity.

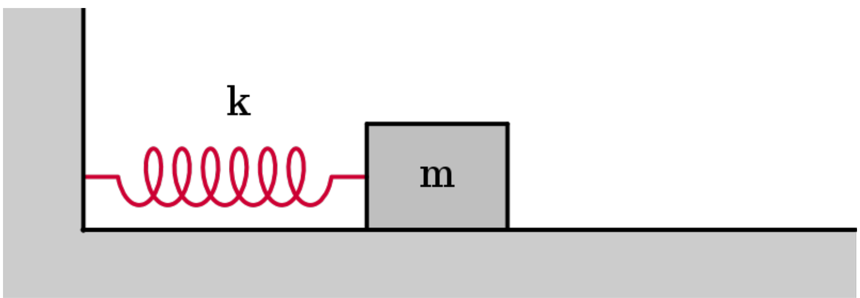

Figure 4.8: A mass attached to a spring executing simple harmonic oscillations

The kinetic and potential energies of the oscillator can be taken as

$$T = \frac{1}{2}m\dot{x}^2 \qquad\qquad U = \frac{1}{2}kx^2$$

Then the Lagrangian of the harmonic oscillator will be

$$L = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

The derivative of the Lagrangian with respect to the generalized coordinate and velocity would be

$$\frac{\partial L}{\partial x} = m\dot{x} \qquad\qquad \frac{\partial L}{\partial \dot{x}} = -kx$$

Now consider the general form of the Lagrangian equation of motion,

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_i}\Big) - \frac{\partial L}{\partial q_i} = 0 \qquad \text{with } q_i = x$$

Then,

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{x}}\Big) - \frac{\partial L}{\partial x} = 0 \qquad \Longrightarrow \qquad \frac{d}{dt}(m\dot{x}) - (-kx) = 0$$

This gives us the well known harmonic oscillator equation

$$m\ddot{x} + kx = 0$$

The solution of the above equation can be taken as

$$x = A\sin\omega t \text{ with } \omega = \sqrt{\frac{k}{m}} \text{ and } A \text{ being the amplitude of the motion}$$

4.3.4 Simple pendulum

Consider a simple pendulum constructed using a bob of mass m attached to a inextensible string of length l suspended from a rigid support. The bob can execute oscillations about a mean position due to the action of gravity with acceleration due to gravity g. Let θ_{max} be the maximum angular displacement that will result the bob to reach a height of h with respect to the mean position. Let the instantaneous angular displacement θ be chosen as generalized coordinate and $\dot{\theta}$ be the corresponding generalized velocity. Then the expression for kinetic and potential energies will be

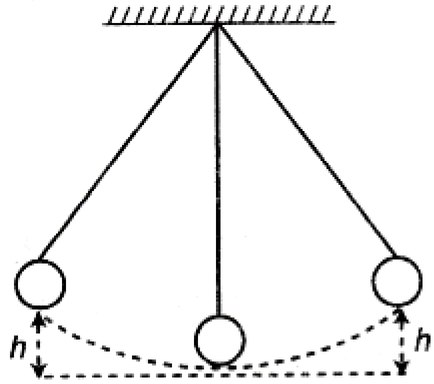

Figure 4.9: Simple pendulum

$$T = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\dot{\theta}^2$$

$$V = mgh = mg(l - l\cos\theta)$$

The Lagrangian of the pendulum will be

$$L = T - V = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\dot{\theta}^2 - mg(l - l\cos\theta)$$

Let us find the derivatives of the Lagrangian with respect to the generalized coordinate and velocity.

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta \qquad \text{and} \qquad \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

Using the above derivatives in the Lagrangian equation of motion as

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_i}\Big) - \frac{\partial L}{\partial q_i} = 0 \qquad \text{with } q_i = \theta$$

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{\theta}}\Big) - \frac{\partial L}{\partial \theta} = 0 \qquad \Longrightarrow \frac{d}{dt}(ml^2\dot{\theta}) + mgl\sin\theta = 0$$

The above equation of motion can be simplified as,

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

Under oscillations of small amplitude, the angular displacement will be very small and then we can approximate $\sin\theta \approx \theta$. Then the equation of motion reduces to equation of simple harmonic motion with a frequency $\omega = \sqrt{\frac{g}{l}}$.

$$\ddot{\theta} + \omega^2\theta = 0$$

4.3.5 Atwood machine

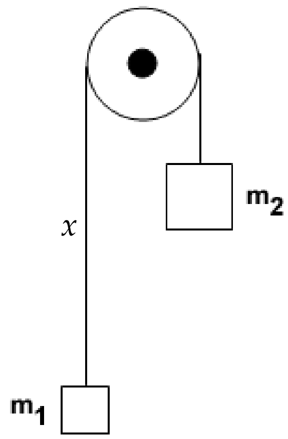

Figure 4.10: Atwood machine

Atwood's machine is an illustration of a simple mechanical system with a holonomic constraint. It consists of two masses m_1 and m_2 tied together by means of a light inextensible cord of length l. The cord passes round a light frictionless pulley and the two masses hang on the two sides of the pulley. We can observe from the figure below, that there is only one variable x, since the length of the cord fixed. The kinetic and potential energies of the system are given by

$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2$ Here, the velocity of mass m_2 is the derivative of $l - x$ that is $\dot{x}$

$$V = -m_1gx - m_2g(l-x)$$

The potential energy is negative because the reference is taken at pulley, and the masses are below it.

Then the Lagrangian of the system will be

$$L = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2 + m_1gx + m_2g(l-x)$$

The derivatives of the Lagrangian with respect to the generalized coordinate and velocity will be

$$\frac{\partial L}{\partial x} = (m_1 - m_2)g \qquad \frac{\partial L}{\partial \dot{x}} = (m_1 + m_2)\dot{x}$$

Consider the general Lagrangian equation of motion,

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_i}\Big) - \frac{\partial L}{\partial q_i} = 0 \qquad \text{with } q_i = x$$

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{x}}\Big) - \frac{\partial L}{\partial x} = 0 \qquad \Longrightarrow \quad \frac{d}{dt}[(m_1 + m_2)\dot{x}] - (m_1 - m_2)g = 0$$

$$\ddot{x} = \frac{m_1 - m_2}{m_1 + m_2} g$$

On integrating the above equation twice we can write the equation of motion in algebraic form as

$$x(t) = x_o + v_o t + \frac{1}{2}\Big(\frac{m_1 - m_2}{m_1 + m_2}\Big) g t^2$$

4.4 Conservation theorems and symmetry in Lagrangian

So far we have discussed Lagrange's dynamical equations of motion. If the system under consideration has n degrees of freedom, we get n second order differential equations. The solution of each equation will need the evaluation of a double integral and hence involve two constants of integrations. These are usually the initial position q_o and initial velocity $\dot{q}_o$. Naturally the solutions of n differential equations will involve $2n$ constants.

In many problems, the solution cannot be obtained in terms of known functions. Moreover, sometimes solutions of the type $q_j = q_j(t)$ are no interest to us. For example, when we study the motion of systems consisting of atoms and molecules we are only interested in evaluation of quantities such as energies and momenta. However, information regarding the physical nature of the motion of the system can often be extracted without integrating the equations of motion.

On considering the symmetries of the system, one can immediately obtain first integrals of the equations of motion. The first integrals are constants of motion. These are the first order differential equations of the type

$$f(q_1, q_2, ..., q_n; \dot{q}_1, \dot{q}_2), ..., \dot{q}_n, t) = constant$$

It is obvious that the first integral contains the first derivatives of q's. The first integrals reveal a lot of information regarding the system under consideration.

Consider a system of particles in a conservative force field. Then the potential energy V depends only upon the position and we have,

$$\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{x}_i} = \frac{\partial (T-V)}{\partial \dot{x}_i} = \frac{\partial}{\partial \dot{x}_i} \sum_j \frac{1}{2} m_j (\dot{x}_j^2 + \dot{y}_j^2 + \dot{z}_j^2) = m_i \dot{x}_i = p_{x_i}$$

In above equation, p_{x_i} is the momentum of i^{th} along x_i axis. We can generalize this result and define the generalized momentum as

$$p_j = \frac{\partial L}{\partial \dot{q}_j}$$

This is often also called canonical momentum or conjugated momentum.

Now consider the Lagrangian equation of motion,

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_j}\Big) - \frac{\partial L}{\partial q_j} = 0$$

Substituting the expression for canonical momentum in above equation we can write

$$\frac{dp_j}{dt} - \frac{\partial L}{\partial q_j} = 0 \qquad \Longrightarrow \qquad \frac{dp_j}{dt} = \frac{\partial L}{\partial q_j}$$

In case any one or more generalized coordinates are absent in the Lagrangian even though the corresponding generalized velocities are present, the derivative of the Lagrangian with respect to those generalized coordinates becomes zero. Then the time derivative of the corresponding canonical momentum also becomes zero. This results in the constancy of the canonical momentum.

Thus we can conclude that if any one or more generalized coordinates are absent in the Lagrangian, then the corresponding canonical momentum gets conserved. Such coordinates are called cyclic or ignorable coordinates.

If a generalized coordinate representing a translation motion is absent in the Lagrangian, then the corresponding linear momentum gets conserved. Here we can remember the Lagrangian of a free particle in cartesian coordinate system.

$$L = \frac{1}{2}m\left[\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right]$$

Here, all the three coordinates are absent in the Lagrangian. Hence we don't have to find the equations of motion, we can directly conclude that the canonical momenta corresponding to all the three directions are constant and proceed from there to find the equations of motion.

If a generalized coordinate representing an angular motion is absent in the Lagrangian, then the corresponding angular momentum is conserved. Here we can remember the Lagrangian of a free particle in spherical polar coordinate system.

$$L = T - V = \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\dot{\phi}^2\right]$$

In the above Lagrangian, the azimuthal angular velocity $\dot{\phi}$ is present, but ϕ is absent in the Lagrangian. Hence the corresponding azimuthal angular momentum is a constant of motion.

If the Lagrangian is not an explicit function of time, then the total energy of the system is conserved. This is also known as homogeneity of time.

$$\frac{\partial L}{\partial t} = 0 \qquad \Longrightarrow \qquad \text{Energy is conserved}$$

4.5 Check your progress

Check your progress by answering the questions below.

1. Define Lagrangian.
2. What is a free particle.
3. Write the Lagrangian of simple harmonic oscillator.
4. On what factors the acceleration in Atwood machine depends?
5. What are cyclic coordinates.
6. What is homogeneity of time? What is its significance?

4.6 Keywords

- Lagrangian

- Free particle
- Harmonic oscillator
- Simple pendulum
- Atwood machine
- Cyclic or ignorable coordinates

4.7 Worked examples

1. A body of mass m is thrown as projectile. Construct the Lagrangian and determine the equations of motion.

   Answer:

   Data:

   Let us assume the constant gravitational field acts down the z-axis and for simplicity let us consider the plane of projectile is in xz-plane. Then let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be its position vector and $\dot{\vec{r}} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}$ be its velocity. Because the projectile is moving in xz-plane, the y-coordinate of the particle is zero. As no motion along y-axis, the corresponding velocity becomes zero.

   Then the expression for the kinetic and potential energies of the body will be

   $$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{z}^2) \qquad \text{and} \qquad V = -mgh = -mgz$$

   Then the Lagrangian of the system will be

   $$L = T - V = \frac{1}{2}m(\dot{x}^2 + \dot{z}^2) + mgz$$

   In the above expression, note that the coordinate x is absent. This means it is cyclic and corresponding conjugate momentum is constant of motion.

   $$\frac{\partial L}{\partial \dot{x}} = m\dot{x} = C_1 \text{ a constant}$$

   Hence let us only find the derivative of the Lagrangian with respect to the other coordinate.

$$\frac{\partial L}{\partial z} = mg \qquad \text{and} \qquad \frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

Now consider the Lagrangian equation of motion

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{z}}\Big) - \frac{\partial L}{\partial z} = 0 \qquad \Longrightarrow \qquad \frac{d}{dt}\Big(m\dot{z}\Big) - mg = 0$$

$\Longrightarrow$

$$m\ddot{z} = mg$$

Thus the equations of motion in differential equation form would be

$$m\dot{x} = C_1 \qquad \text{and} \qquad m\ddot{z} = mg$$

2. Masses m and $2m$ are connected by a light inextensible string which passes over a pulley of mass $2m$ and radius a. Write the Lagrangian and find the acceleration of the system.

Answer:

The given system is an Atwood machine with only one degree of freedom. Let x be the distance of mass m from the pulley and is chosen as the generalized coordinate. Let l be the length of the string and the centre of the pulley is taken as zero for potential energy.

The kinetic energy of the system would be

$$T = \frac{1}{2}m\dot{x}^2 + m\dot{x}^2 + \frac{1}{2}I\omega^2 = \frac{3}{2}m\dot{x}^2 + \frac{1}{2}I\Big(\frac{\dot{x}}{a}\Big)^2 = \frac{1}{2}\Big(3m + \frac{I}{a^2}\Big)\dot{x}^2$$

The potential energy would be

$$V = -mgx - 2mg(l - x)$$

Then the Lagrangian of the system will be

$$L = \frac{1}{2}\Big(3m + \frac{I}{a^2}\Big)\dot{x}^2 + mgx + 2mg(l - x)$$

Lets find the derivatives of the Lagrangian,

$$\frac{\partial L}{\partial x} = -mg \qquad\qquad \frac{\partial L}{\partial \dot{x}} = \left(3m + \frac{I}{a^2}\right)\dot{x}$$

Then the equation of motion of the system will be

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \qquad \Longrightarrow \qquad \left(3m + \frac{I}{a^2}\right)\ddot{x} + mg = 0$$

The expression for acceleration will be

$$\ddot{x} = -\frac{mg}{\left(3m + \frac{I}{a^2}\right)}$$

If we assume the pulley is in the form of a disc, then the moment of inertia will be $\frac{ma^2}{2}$. Then the acceleration will be $-g/4$.

3. Find the equations of motion of an LC circuit using Lagrangian formulation.

   Answer: Consider an LC circuit containing an inductor and a capacitor in series. Let q be the charge on the plates of capacitor and let $\dot{q}$ be the current through the circuit.

   As the capacitor stores the charges and hence the energy stored in the capacitor can be taken for potential energy as

$$V = \frac{q^2}{2C}$$

   Similarly the energy held by the inductor is due to the flow of charges. Hence, the energy of the inductor can be treated equivalent to the kinetic energy of the system. Lower case l is used to represent inductance to avoid confusion between the Lagrangian and inductance.

$$T = \frac{1}{2}l\dot{q}^2$$

   Then the Lagrangian of the system will be

$$L = T - V = \frac{1}{2}l\dot{q}^2 - \frac{q^2}{2C}$$

Now consider the Lagrangian equation of motion and use the above Lagrangian in it to obtain the equations of motion.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \qquad \Longrightarrow \qquad \frac{d}{dt}(l\dot{q}) - \frac{q}{C} = 0$$

$\Longrightarrow$

$$\ddot{q} - \frac{1}{C}q = 0 \qquad \Longrightarrow \qquad \ddot{q} - \omega^2 q = 0$$

Where $\omega = \dfrac{1}{\sqrt{lC}}$. The above equation is a simple harmonic equation and hence indicate that the current as well as the charge on the plates of capacitor oscillate with a frequency of $\omega = \dfrac{1}{\sqrt{lC}}$.

4.8 Questions for self study

1. Explain the procedure to find the equations of motion using Lagrangian formulation
2. Determine the equations of motion of a free particle in cartesian coordinate system.
3. Determine the equations of motion of a free particle in spherical polar coordinate system.
4. Determine the equations of motion of simple harmonic oscillator.
5. Determine the equations of motion of simple pendulum.
6. Obtain an expression for acceleration of both the masses in Atwood machine.
7. Write a note on conservation theorems and symmetry in Lagrangian.

4.9 Answers to check your progress

1. Lagrangian is a function that is difference between the kinetic energy and potential energy functions.
2. A particle on which no forces are acting is called a free particle. In other words, a particle for which the potential energy is zero is called a free particle.
3. $L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$

4. The acceleration in Atwood machine depends on the acceleration due to gravity and the masses used.
5. Coordinates that are absent in Lagrangian are called cyclic coordinates.
6. Independence of Lagrangian of a system over time is called homogeneity of time. This leads to the conservation of energy of the system.

4.10 References

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